The Bois Marie Algebraic Geometry Seminar

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Topos Theory And
Etale Cohomology of Schemes

(SGA 4)

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TOPOS THEORY
(Lecture IV, parts 0-4 "TOPOS")

Lecture IV

Topos

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0 Introduction

0.1 In Lecture II we saw various exactness properties of categories of the form $\widetilde{\mathcal{C}}$ = the category of the sheaves of sets on $\mathcal{C}$, where $\mathcal{C}$ is a small site, properties which can be expressed by stating that, in many respects, these categories (which we will call *toposes*) inherit the familiar properties of the category $\mathbf{Set}$ of (small) sets. On the other hand, experience teaches that there are grounds to consider various situations in Mathematics *primarily as a technique for constructing corresponding categories of sheaves* (of sets), i.e. *the corresponding "toposes"*. Apparently, all the genuinely important concepts linked to a site (for example its cohomology invariants, studied in Lecture V, various other "topological" invariants, such as its homotopy invariants studied recently by M. Artin and B. Mazur [1] and the notions studied in J. Giraud's book on non-commutative cohomology) are in fact directly expressed in terms of the associated topos. From this perspective, it is convenient to see two sites as being essentially equivalent when the associated toposes are equivalent categories, and to consider that the concept of a site (at least in the case, of great importance in practice, where its topology is less fine than its canonical topology) amounts to that of a topos $\mathcal{E}$ (namely the associated topos, made up of the sheaves of sets on the site), and of a generating family

300 of elements of $\mathcal{E}$ (cf. II 4.9, and 1.2.1 above). This standpoint is analogous to that which consists of associating a group with a system of generators and a system of relationships between these generators, and being interested more in the structure of these groups than the system of generators and relationships which served to give rise to it (considered as secondary data of the situation). Moreover, the "comparison lemma" III 5.1. provides numerous examples of pairs of sites $\mathcal{C}$, $\mathcal{C}'$ which are non-isomorphic, and even non-equivalent as categories, which give rise to equivalent toposes, such that there are grounds to consider $\mathcal{C}$ and $\mathcal{C}'$ as essentially equivalent.

0.2 In this lecture, we give a characterisation of toposes using the simple exactness properties (of J. Giraud), we study the natural notion of topos morphism, inspired by the notion of a continuous map from one topological space to another, and we develop, in the framework of toposes, a number of constructions familiar from conventional sheaf theory (***Hom*** sheaves, tensor product sheaves, supports). Finally, we will show (following M. Artin) how a topos can be reconstituted from an "open" subtopos of itself, from its "closed" complement, and from a certain left exact functor that links them, which can, moreover, be chosen more or less arbitrarily.

0.3 We have here a procedure for the gluing of toposes which, applied to toposes coming from ordinary topological spaces, will give, in general, a topos which will no longer be of the same type. This is a first indication of the remarkable stability of the notion of topos under various natural constructions, which the notion of
301 topological space (that inspired the notion of topos) lacks. For a second notable example, we also point to that of classifying topos relative to a group of a topos (c.f. [1] or 5.9 below), inspired by the classical notion of classifying space of a topological group, and the notion of "modular topos" associated with various "module problems" in Algebraic geometry and Analytical geometry [10] [13].

Other toposes, such as the *étale topos* of a scheme (systematically studied in this Seminar, from Lecture VII onwards) or the *crystalline topos* of a relative scheme [6] enter in naturally when one wants to develop usable cohomology theories for abstract algebraic varieties (and more generally for schemes), which replace the classic Betti cohomology of algebraic varieties over the complex field.

0.4 Therefore, it can be said that the notion of topos, natural derivative of the *sheaf standpoint* in Topology, constitutes in turn a substantial enlargement of the notion of topological space[1], including a large number of situations which were not formerly considered as arising from topological intuition. The characteristic mark of such situations is that there is a notion of "localisation", a notion which is formalised precisely by the notion of site and, all things considered, by that of topos (via the site's associated topos). Exactly as the term "topos" itself is supposed to suggest, it seems reasonable and legitimate to the authors of the current Seminar to consider that the object of Topology is the study of *toposes* (and not merely topological spaces).

0.5 It seemed useful to us to include in this general lecture on toposes a fairly
302 large number of examples, many of which have only a distant relationship with the proposed initial goal of this seminar (i.e. the study of étale cohomology). The hurried reader, if only interested in étale cohomology, could easily skip these examples, and moreover most of the present lecture, to which he can refer back if needed.

1 Definition and characterisation of toposes

Definition 1.1 *A category $\mathcal{E}$ is called a $\mathscr{U}$-*topos, *or simply* topos *if their is no risk of confusion, if there exists a site $\mathcal{C} \in \mathscr{U}$ such that $\mathcal{E}$ is equivalent to the category $\widetilde{\mathcal{C}}$ of $\mathscr{U}$-sheaves of sets on $\mathcal{C}$.*

1.1.1 Let $\mathcal{E}$ be a $\mathscr{U}$-topos. We always deem $\mathcal{E}$ to be equipped with its canonical topology (II 2.5), which makes it a *site*, and even, by virtue of 1.1.2 d) below, a $\mathscr{U}$-site (II 3.0.2). Unless expressly stated otherwise, we will not consider any topology on $\mathcal{E}$ other than the one just described.

1.1.2 It was seen in II 4.8, 4.11 that a $\mathscr{U}$-topos $\mathcal{E}$ is a $\mathscr{U}$-category (I 1.1) satisfying the following conditions:

a) Projective finite limits are representable in $\mathcal{E}$.

b) Direct sums indexed by an element of $\mathscr{U}$ are representable in $\mathcal{E}$. They are disjoint and universal (II 4.5).

[1] Cf. [9], or 4.1 and 4.2 below, for the precise relationship between the notion of topos and that of topological space.

c) Equivalence relations in $\mathcal{E}$ are universal effective (I 10.6).

303 d) $\mathcal{E}$ has a generating family (II 4.9) indexed by an element of $\mathcal{U}$.

In fact, we will see that these intrinsic properties *characterise* $\mathcal{U}$-toposes:

Theorem 1.2 *(J. Giraud). Given $\mathcal{E}$ a $\mathcal{U}$-category. The following properties are equivalent:*

i) $\mathcal{E}$ is an $\mathcal{U}$-topos (1.1).

ii) $\mathcal{E}$ satisfies conditions a), b), c) and d) of 1.1.2.

iii) $\mathcal{U}$-sheaves on $\mathcal{E}$ for the canonical topology are representable, and $\mathcal{E}$ has a small generating family (condition 1.1.2 d)).

iv) There exists a category $\mathcal{C} \in \mathcal{U}$ and a fully faithful functor $i : \mathcal{E} \longrightarrow \hat{\mathcal{C}}$ (where $\hat{\mathcal{C}}$ represents the category of $\mathcal{U}$-presheaves on $\mathcal{C}$) having a left adjoint functor a which is left exact.

i') There exists a site $\mathcal{C} \in \mathcal{U}$, such that projective limits are representable in $\mathcal{C}$ with a topology on $\mathcal{C}$ that is less fine than its canonical topology (II 2.5) and such that $\mathcal{E}$ is equivalent to the category $\widetilde{\mathcal{C}}$ of $\mathcal{U}$-sheaves of sets on $\mathcal{C}$.

Furthermore:

Corollary 1.2.1 *Let $\mathcal{E}$ be a $\mathcal{U}$-topos, $\mathcal{C}$ a full subcategory of $\mathcal{E}$ provided with the topology induced (III 3.1) by that of $\mathcal{E}$ (1.1.1). Consider the functor*

$$\mathcal{E} \longrightarrow \widetilde{\mathcal{C}}$$

which associates with all $X \in Ob(\mathcal{E})$ the restriction to $\mathcal{C}$ of the functor represented by X. This functor is an equivalence of categories if and only if $Ob(\mathcal{C})$ is a generating family of $\mathcal{E}$.

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The equivalence i) $\Longleftrightarrow$ iv) results directly from II 5.5, and it has already been recalled above that i) $\Longrightarrow$ ii). As i') $\Longrightarrow$ i) trivially, ii) $\Longrightarrow$ iii) and iii) $\Longrightarrow$ i') still need to be proved, which will be done in 1.2.4 and 1.2.3 below. The corollary is thus obtained by noticing that the functor envisaged factorizes as $\mathcal{E} \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \widetilde{\mathcal{C}}$, with the result that, $\mathcal{E} \longrightarrow \widetilde{\mathcal{E}}$ being an equivalence due to iii), the question comes back to that of determining when $\widetilde{\mathcal{E}} \longrightarrow \widetilde{\mathcal{C}}$ is an equivalence. The conclusion then follows thanks to the "comparison lemma" III 4.1, cf. proof 1.2.3 below.

Proof 1.2.3 Proof of iii) $\Longrightarrow$ i'). Let $\mathscr{X} = (X_i)_{i\in I}$, $I \in \mathscr{U}$, be a small generating family of $\mathcal{E}$. As $\mathcal{E}$ is a $\mathscr{U}$-category, the set of isomorphic classes of finite diagrams in $\mathcal{E}$ whose objects are the elements X_i is $\mathscr{U}$-small. Consequently, the smallest set $\mathscr{X}'$ of objects of $\mathcal{E}$, containing the finite projective limits of objects of $\mathscr{X}$, is a countable union of small sets and is thus small [2]. Setting $\mathscr{X}^{(n+1)} = \mathscr{X}^{(n)(1)}$, and $\mathscr{X} = \bigcup_n \mathscr{X}^{(n)}$, it can be assumed, even if it entails increasing the family of generators, that $\mathscr{X}$ is closed under finite projective limits. Let $\mathscr{V}$ be a universe containing $\mathscr{U}$ such that $\mathcal{E}$ is $\mathscr{V}$-small. For every object H of $\mathcal{E}$, let us designate $I(H)$ the set $\coprod_{i\in I} \mathbf{Hom}(X_i, H)$. Let H be an object of $\mathcal{E}$ and $H' \hookrightarrow H$ the subsheaf of H, for the canonical topology, that is the "union" of the images of the morphisms $u : X_i \longrightarrow H$, $(u, i) \in I(H)$ (II 4.1). As H' is a subsheaf of a $\mathscr{U}$-sheaf, H' is a $\mathscr{U}$-sheaf. It is, therefore, representable. As the family $\mathscr{X}$ is generating and as for all $i \in I$, the map $\mathbf{Hom}(X_i, H') \longrightarrow \mathbf{Hom}(X_i, H)$ is bijective, the morphism $H' \hookrightarrow H$ is an isomorphism (II 4.9). Consequently (II 6.1), the family $(u : X_i \longrightarrow H)$, $(u, i) \in I(H)$ is covering for the canonical topology of $\mathcal{E}$.

305 Therefore, every object of $\mathcal{E}$ can be covered, for the canonical topology of $\mathcal{E}$, by the objects of $\mathscr{X}$. Let $\mathcal{C}$ be the full subcategory of $\mathcal{E}$ defined by the objects of $\mathscr{X}$ and $u : \mathcal{C} \hookrightarrow \mathcal{E}$ the inclusion functor. The topology $\mathscr{C}$ induced on $\mathcal{C}$ by the canonical topology of $\mathcal{E}$ is less fine than the canonical topology of $\mathcal{C}$, and when $\mathscr{U}$ has an infinite cardinal element, then finite projective limits are representable in $\mathcal{C}$. Therefore, from III 5.1, the functor $F \longmapsto F \circ u$ is an equivalence between $\mathcal{E}$ and the category of $\mathscr{U}$-sheaves on $\mathcal{C}$ for the topology $\mathscr{C}$, Q.E.D.

Proof 1.2.4 Proof of ii) $\Longrightarrow$ iii). This proof has four steps.

Let $\mathscr{V}$ be a universe containing $\mathscr{U}$, such that $\mathcal{E}$ is an element of $\mathscr{V}$. Let $\widetilde{\mathcal{E}}$ be the category of $\mathscr{V}$-sheaves on $\mathcal{E}$ for the canonical topology, and $J_{\mathcal{E}} : \mathcal{E} \longrightarrow \widetilde{\mathcal{E}}$ the canonical functor.

1.2.4.1 Let $(g_i : G_i \longrightarrow H)$, $i \in I \in \mathscr{U}$ be an epimorphic family of $\widetilde{\mathcal{E}}$. If the G_i and the fibred products $G_i \times_H G_j$ are representable, then the sheaf H is representable.

In fact, the direct sum $\coprod_{i\in I} G_i$ is representable in $\mathcal{E}$ (II 4.1) by a repre-

[2]We are thinking here about the classes of objects of $\mathcal{E}$ up to isomorphism.

sentable sheaf G (property b) and II 4.6). The same holds for the direct sum

$$K \;=\; \coprod_{(i,j)\,\in\, I\,\times\, I} G_i \,\times_H\, G_j\ .$$

Furthermore, the diagram $K \;\rightrightarrows\; G \;\longrightarrow\; H$ is exact and K is the fibred square of G above H. This last property is true in the category of presheaves, thus true in the category of sheaves, II 4.1. From this it is deduced, following c) and II 4.7, that the sheaf H is representable.

1.2.4.2 Let $X_\alpha\ ,\ \alpha \,\in\, A \,\in\, \mathscr{U}$ be a family of generators of $\mathcal{E}$. For every sheaf H, let us designate by $I(H)$ the set $\coprod_{\alpha\,\in\,A} \mathbf{Hom}_{\mathcal{E}}(X_\alpha\ ,\ H)$. The family
306 $(u\ :\ X_\alpha\ \longrightarrow\ H,\ (u,\alpha)\ \in\ I(H))$ is epimorphic in $\widetilde{\mathcal{E}}$. When H is a $\mathscr{U}$-sheaf, $I(H)$ is an element of $\mathscr{U}$.

In fact, as every sheaf H is the target of an epimorphic family of morphisms whose sources are representable sheaves (I 3.4 and II 4.1), it is sufficient to demonstrate the first assertion when the sheaf H is representable. So, let G be the image of the family $(u\ :\ X_\alpha\ \longrightarrow\ H,\ (u,\alpha) \in I(H))$. The morphism $G\ \longrightarrow\ H$ is a monomorphism. Consequently, this is the same situation as the first step, because $X_\alpha \times_G X_\beta$ is isomorphic to $X_\alpha \times_H X_\beta$ which is representable, and, furthermore, $I(H)$ is an element of $\mathscr{U}$. Thus it is deduced that G is representable. Because X_α is a generating family, $G\ \longrightarrow\ H$ is an isomorphism. The last assertion is obvious from the definition of $\mathscr{U}$-sheaves.

1.2.4.3 Every subsheaf of a representable sheaf is representable.

So, let $G\ \longrightarrow\ H$ be a subsheaf of a representable sheaf. Thus, G is a $\mathscr{U}$-sheaf. The family $(u\ :\ X_\alpha\ \longrightarrow\ G,\ (u,\alpha) \in I(G))$ is thus epimorphic in $\widetilde{\mathcal{E}}$ and indexed by an element of $\mathscr{U}$. Furthermore, the fibred products $X_\alpha\ \times_G\ X_\beta$ are isomorphic to the fibred products $X_\alpha\ \times_H\ X_\beta$ which are representable. Consequently, this is the same situation as in the first step and G is representable.

1.2.4.4. Every $\mathscr{U}$-sheaf is representable.

In fact, by virtue of the first and second steps, it is sufficient to demonstrate that the fibred products $X_\alpha\ \times_H\ X_\beta$ are representable. Now, these fibred products are the subobjects of the products $X_\alpha\ \times\ X_\beta$. Thus, the third step is the final one. This completes the proof of theorem 1.2.

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Remark 1.3 Of course, for a given $\mathscr{U}$-topos $\mathcal{E}$, there is not, in general, a preferred manner of representing it up to equivalence in the form $\widetilde{\mathcal{C}}$, where $\mathcal{C}$ is a small site; or, which essentially comes down to the same thing due to 1.2.1 when we limit ourselves to those $\mathcal{C}$ whose topology is less fine than the canonical topology, there is not a preferred *small* generating family in $\mathcal{E}$. When the condition that $\mathcal{C}$ is small is no longer imposed, there is on the other hand (by virtue of 1.2 iii)) an entirely canonical choice of a $\mathscr{U}$-site such that $\mathcal{E}$ is equivalent to $\widetilde{\mathcal{C}}$, namely $\mathcal{E}$ itself! This is one of the technical reasons why it is not useful in practice to work only with small sites: in fact, the most important sites of all, namely $\mathscr{U}$-toposes, are not small! Furthermore, the topos generating sites which enter into many questions in algebraic geometry (indeed in topology, cf. 2.5) are not small either; e.g.: the étale site of a scheme (VIII 1).

Proposition 1.4 *Let $\mathcal{E}$ be a $\mathscr{U}$-topos, $\mathcal{E}'$ a category (not necessarily a $\mathscr{U}$-category), F a presheaf on $\mathcal{E}$ with values in $\mathcal{E}'$. For F to be a sheaf with values in $\mathcal{E}'$ (II 6.1), it is necessary and sufficient that F transforms inductive $\mathscr{U}$-limits in $\mathcal{E}$ into projective limits in $\mathcal{E}'$.*

Let $\mathscr{V}$ be a universe containing $\mathscr{U}$ such that $\mathcal{E}'$ is a $\mathscr{V}$-category. Composing F with the functors $\mathbf{Hom}(X', -) : \mathcal{E}' \longrightarrow \mathscr{V} - \mathbf{Set}$ defined by the objects X' and $\mathcal{E}'$, leads back to the case where $\mathcal{E}' = \mathscr{V} - \mathbf{Set}$, where $\mathscr{V}$ is a universe. Let us assume that F transforms inductive $\mathscr{U}$-limits into projective limits, thus it
308 follows directly from the definitions that F is a sheaf, because a family covering $X_i \longrightarrow X$ in $\mathcal{E}$, by the definition of the canonical topology of $\mathcal{E}$, allows X to be considered as an inductive limit of the diagram $X_i \times_X X_j \begin{smallmatrix} \nearrow & X_i \\ \searrow & X_j \end{smallmatrix}$. Let us assume F is a sheaf, and let us prove that it transforms inductive $\mathscr{U}$-limits into projective limits. If $\mathscr{V} \subset \mathscr{U}$, it can be assumed that $\mathscr{V} = \mathscr{U}$ and it is sufficient to apply the criteria 1.2 iii). To deal with the general case, a little more work is needed. Let $\mathcal{C}$ be a full subcategory of $\mathcal{E}$ produced by a generating family indexed by an element of $\mathscr{U}$, and let $u : \mathcal{C} \longrightarrow \mathcal{E}$ be the inclusion functor. Let us provide $\mathcal{C}$ with the topology induced by the canonical topology of $\mathcal{E}$ (III 4.5). Denote by $\widetilde{\mathcal{C}}_{\mathscr{U}}, \widetilde{\mathcal{C}}_{\mathscr{V}}$ the categories of sheaves on $\mathcal{C}$, $\widetilde{\mathcal{E}}_{\mathscr{V}}$ the category of $\mathscr{V}$-sheaves on $\mathcal{E}$ for the canonical topology, $J_{\mathcal{E}} : \mathcal{E} \longrightarrow \widetilde{\mathcal{E}}_{\mathscr{V}}$ the canonical functor and $i_{\mathscr{U},\mathscr{V}} : \widetilde{\mathcal{C}}_{\mathscr{U}} \longrightarrow \widetilde{\mathcal{C}}_{\mathscr{V}}$ the inclusion functor. The diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{C}}_{\mathscr{U}} & \xrightarrow{i_{\mathscr{U},\mathscr{V}}} & \widetilde{\mathcal{C}}_{\mathscr{V}} \\ \uparrow & & \uparrow \\ \mathcal{E} & \xrightarrow{J_{\mathcal{E}}} & \widetilde{\mathcal{E}}_{\mathscr{V}} \end{array}$$

where the vertical arrows are induced by the functor $F \mapsto F \circ u$, is commutative. Furthermore, it follows from the explicit construction of the associated sheaf functor (II 3) and from II 4.1 that the functor $i_{\mathscr{U},\mathscr{V}}$ commutes with inductive $\mathscr{U}$-limits. Furthermore, it results from III 5.1 and from 1.5 that the vertical arrows of the above diagram are equivalences of categories. Consequently, $J_{\mathcal{E}} : \mathcal{E} \longrightarrow \widetilde{\mathcal{E}}_{\mathscr{V}}$ commutes with inductive $\mathscr{U}$-limits. For every object X of $\mathcal{E}$, we have:

$$F(X) \cong \mathbf{Hom}_{\widetilde{\mathcal{E}_{\mathscr{V}}}}(J_{\mathcal{E}}(X), F).$$

309 Thus F transforms the inductive $\mathscr{U}$-limits of $\mathcal{E}$ into projective limits.

Corollary 1.5 *Let $\mathcal{E}$ be a $\mathscr{U}$-topos, $\mathcal{E}'$ a $\mathscr{U}$-category, $f : \mathcal{E} \longrightarrow \mathcal{E}'$ a functor. For f to have a right adjoint, it is necessary and sufficient that f commutes with inductive $\mathscr{U}$-limits.*

This is a direct consequence of 1.4 for the case $\mathcal{E}' = \mathscr{U} - \mathbf{Set}$.

Corollary 1.6 *Let $\mathcal{E}$, $\mathcal{E}'$ be two $\mathscr{U}$-toposes, $f : \mathcal{E} \longrightarrow \mathcal{E}'$ a functor. The following conditions are equivalent:*

i) *f commutes with inductive $\mathscr{U}$-limits.*

ii) *f has a right adjoint.*

iii) *f is continuous (III 1.1).*

The equivalence i) $\Longleftrightarrow$ ii) has been seen in 1.5. To prove i) $\Longleftrightarrow$ iii), let us apply the definition of continuous functors, choosing a universe $\mathscr{V}$ such that $\mathscr{U} \in \mathscr{V}$ (thus, $\mathcal{E}$,$\mathcal{E}'$ are $\mathscr{V}$-small sites). It needs to be shown that for every $\mathscr{V}$-sheaf F on $\mathcal{E}'$, the composite $F \circ f$ is a sheaf on $\mathcal{E}$, i.e. by (1.4) it transforms inductive $\mathscr{U}$-limits into projective limits. It is sufficient for this to have i), because by virtue of 1.4 F itself transforms inductive $\mathscr{U}$-limits into projective limits; it is also necessary, as is seen by taking F to be a representable functor.

Corollary 1.7 *Using the notation from 1.6, for f to be the inverse image functor u^* for a topos morphism (3.1) $u : \mathcal{E}' \longrightarrow \mathcal{E}$, it is necessary and sufficient that f is left exact and transforms epimorphic families into epimorphic families.*

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The necessity is obvious by definition (N.B. any right exact functor transforms epimorphisms into epimorphisms). The sufficiency results from III 2.6, which implies that f is continuous under the conditions given, and thus commutes with inductive $\mathscr{U}$-limits by virtue of 1.6.

Remark 1.8 Using the notation from 1.5, it can be shown that f has a left adjoint if and only if it commutes with projective $\mathscr{U}$-limits. In other words, a *covariant* functor $\mathcal{E} \longrightarrow (\mathscr{U} - \mathbf{Set})$ is representable if and only if it commutes with projective $\mathscr{U}$-limits[3]. We only outline the proof here, which has two stages:

a) The standard arguments [5, No. 195, §3] show that if F commutes with $\varprojlim$, it is prorepresentable by a strict projective system $(T_i)_{i\in I}$, where I is a filtered ordered set, *not necessarily small*. It can be assumed that if $i > j$, then $T_i \longrightarrow T_j$ is not an isomorphism, and under this assumption, F is representable if and only if I is small (which implies in fact that the projective system is essentially constant).

b) To prove that I is small, knowing that for every object X of $\mathcal{E}$ the set $F(X) = \varinjlim_i \mathbf{Hom}(T_i, X)$ is small, it is sufficient to have a *small cogenerating family* $(X_j)_{j\in J}$ (i.e. which is generating for the opposite category $\mathcal{E}^\circ$). Now, it will be shown that in a $\mathscr{U}$-topos $\mathcal{E}$ a small cogenerating family always exists.

c) To prove this last point, note that by the standard arguments [4] every object X of $\mathcal{E}$ has a monomorphism into an "injective" object; then for every
311 generating family (L_α) of $\mathcal{E}$, if each L_α is thus immersed into an injective object I_α, the family (I_α) is cogenerating.

[3] A more general statement is found in I 8.12.8, 8.12.9, the proof outline that follows in a) to c) corresponds to the proof given in *loc. cit.*

2 Examples of toposes

2.0 We have gathered together in this section a fairly large number of typical examples of toposes, which the reader will already have had the opportunity to meet from other sources, and which are meant to give easy access to topos "yoga". For other examples of topologies on sites (drawn from algebraic geometry), giving rise to as many toposes, SGA 3 IV 6 may be consulted, and (for the étale topos) Lecture VII of this seminar. As we will hardly ever refer to this section later on, except for notation and terminology, we leave as an exercise for the reader, for his general education, the checking of statements which we have included with these examples. All the examples in this section will be clarified in Section 4 where we will examine how the constructions are functorial, and in the following sections as illustrations of general notions relating to toposes.

2.1 Topos associated with a topological space

Let X be a small topological space, $\mathcal{O}(X)$ the category of open sets of X, provided with the canonical topology (III 2.9.1). We designate by $Top(X)$[4] the topos of $\mathscr{U}$-sheaves on $\mathcal{O}(X)$. This topos is equivalent to the category of étale topological spaces above X, associating with every such space X' on X the sheaf
312 $U \mapsto \Gamma(X'/U) = \mathbf{Hom}_X(U, X')$ on $\mathcal{O}(X)$ [8]. Inevitably, the same letter X will sometimes be used, by abuse of notation, to denote the topos $Top(X)$ defined by the topological space X.

The preceding example is, of course, the one that has largely served as a guide and intuitive support for the development of topos theory. However care must be taken as the toposes deduced from topological spaces are by their nature very special, notably due to the fact that they were described by sites $\mathcal{C} = \mathcal{O}(X)$ where *all morphisms are monomorphisms* (thus whose underlying category amounts to a preordered set). In particular the result of this is that the sheaves represented by the objects of $\mathcal{C}$ are subsheaves of the final sheaf, and consequently that *the subsheaves of the final sheaf form a generating family of the topos under consideration*. This property is not shared by most toposes which come about naturally in geometric algebra or in algebra, cf. below for example. It is more or less characteristic of toposes of the form $Top(X)$ (cf. 7.1.9 below).

It is easily checked that the map $\mathcal{O}(X) \longrightarrow Top(X)$, which associates

[4] *Translator*: More familiar now as $Sh(X)$.

with every open set of X the sheaf that it represents, is a bijection of $\mathcal{O}(X)$ with the set of subobjects of the final object of $Top(X)$, this bijection even being an isomorphism for the natural order structures, i.e. inducing an isomorphism of corresponding categories. This suggests that it should be possible to reconstitute up to homeomorphism the topological space X, when $Top(X)$ is known up to equivalence. We will see below that this is indeed so, in return for a slight restriction on X.

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2.2 Point or final topos, and empty or initial topos

When X is a topological space reduced to a single point, the functor

$$Top(X) \longrightarrow \mathscr{U} - \mathbf{Set}$$
$$F \mapsto F(X)$$

is an equivalence of categories. In particular, this shows that the *category* $\mathscr{U} - \mathbf{Set}$ is a $\mathscr{U}$*-topos*. We have seen in the examples in Lecture II that this $\mathscr{U}$-topos is typical with regards to exactness properties since the verification of many properties (notably the exactness properties) of general toposes leads back to this special topos. The interpretation that we are giving here of the point topological space justifies our abuse of language whereby we call the topos equivalent to the category $\mathscr{U} - \mathbf{Set}$ a *point topos* (even though, as a category, it is not at all equivalent to the point category!). It is the terminology that corresponds to the correct geometric intuition of the role played by these toposes. A point topos is also called, by a similar abuse of language, a *final topos*, cf. 4.3; the topos $\mathscr{U} - \mathbf{Set}$ is called "*the* final topos".

When X is reduced to the empty topological space, then $\mathcal{O}(X)$ is the point category, thus a presheaf F on $\mathcal{O}(X)$ is a sheaf if and only if its value on the single object of $\mathcal{O}(X)$ is a set reduced to a point. It follows that $Top(\phi)$ is isomorphic to the category of $\mathscr{U}$-sets reduced to a point, a category which is equivalent to the point category. From this it is concluded in particular that the point category (as well as every $\mathscr{U}$-category equivalent to this) is a $\mathscr{U}$-topos. It is sometimes called,
314 by abuse of language, the *empty topos* or the *initial topos* (cf. 4.4); it should be noted that it is not equivalent to the empty category.

2.3 Topos associated with an operator space

Let X be a topological space, G a discrete group acting on X by homeomorphisms. The category of G-sheaves on X, or as we also say the sheaves on (X, G), has been defined in [4 5.1]; they are the sheaves (of sets) on X, *provided* with G-actions compatible with those of G on X. It is immediately seen, from Giraud's criteria 1.2 iii), that this category is a topos (N.B. $\mathscr{U}$ is implicit in all of this), which is denoted simply $Top(X, G)$[5]. When G is reduced to be trivial, we are back to example 2.1; when X is reduced to the point space, the topos of sets on which G acts to the left (or $G - \mathbf{Sets}$) is recovered, also called *the classifying topos of the discrete group* G, and denoted B_G. It is easily verified that the only subobject of the final object e of the topos B_G is either e or the empty sheaf ϕ, in particular, if G is trivial, the subobjects of the final object of the classifying topos B_G do not form a generating family of B_G. Thus B_G is not equivalent to a topos of the type $Top(X)$ considered in 2.1.

The notion of G-sheaves was introduced in *loc. cit.* to develop the cohomology theory of Abelian G-sheaves. Interpreting these last as the Abelian sheaves of the topos (X, G), this theory finds itself covered by Lecture V, where
315 it is developed in the framework of general toposes.

It can be considered, more generally, how to attach an appropriate topos to a topological space X, provided with a topological group (not necessarily discrete) G of automorphisms, so as to give rise to an adequate cohomology theory. Similarly in the context of differentiable manifolds, real or complex analysis, or schemes. This is, in fact, possible, cf. 2.5 below.

2.4 The classifying topos of a group

Let $\mathcal{E}$ be a topos, and G a group of $\mathcal{E}$. Let $(\mathcal{E}, G)$ be the category of objects of $\mathcal{E}$ on which G acts. It is immediately seen, by Giraud's criteria, that it is a topos. It is called the *classifying topos* of the group G, and is denoted B_G. When $\mathcal{E}$ is the point topos (2.2) i.e. when G is an ordinary group, we have the classifying topos of 2.3.

The terminology adopted here is justified by the fact that the topos B_G plays a universal role in the classification of "torsors" (or principal homogeneous fibres) under G, or more generally under $G_{\mathcal{E}'} = f^*(G)$, where $\mathcal{E}'$ is a topos "above" $\mathcal{E}$, i.e. provided with a morphism $f : \mathcal{E}' \longrightarrow \mathcal{E}$ (cf. 3.1 below). This role, clarified in [3 Chap V] or in 5.9 below, shows that B_G plays, in the context of toposes, the same role as the classical classifying spaces of topological groups in

[5] *Translator*: Now written $Sh_G(X)$.

the homotopy theory of topological spaces. These last can be seen (cf. 2.5) as a weakened version of the first, obtained by retaining from the classifying topos only the “homotopy type” of the topos in question, in a suitable way which does not need to be specified here.

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2.5 The “large site” and “large topos” of a topological space. The classifying topos of a topological group [6]

Let $\mathscr{U} - \mathbf{Sp}$ or simply $\mathbf{Sp}$ be the category of topological spaces $\in \mathscr{U}$. It is known that in $\mathbf{Sp}$ finite projective limits are representable. Let us consider the pretopology over $\mathbf{Sp}$ (I 1.3) for which $Cov(X)$ is the set of surjective families of open immersions $u_i : X_i \longrightarrow X$. We consider $\mathbf{Sp}$ as a site by means of the topology generated by the preceding pretopology. For every object X of $\mathbf{Sp}$, let us consider the category

$$\mathbf{Sp}/X$$

of objects of $\mathbf{Sp}$ over X, i.e. the topological spaces over X, as a site, thanks to the topology induced by that of $\mathbf{Sp}$ through the forgetful functor $\mathbf{Sp}/X \longrightarrow \mathbf{Sp}$ (III 5.2 4). This site is called the *large site* associated with X. It should be noted that it is not $\in \mathscr{U}$; this is no longer a $\mathscr{U}$-site in the sense of II 3.0.2, and so care is needed in applying the usual results to it. To overcome this inconvenience, a universe $\mathscr{V}$ such that $\mathscr{U} \in \mathscr{V}$ can be chosen, so that $\mathbf{Sp}/X$ becomes a $\mathscr{V}$-site, and one can work with the associated $\mathscr{V}$-topos $\mathbf{Sp}/X$, which will be denoted $TOP(X)$ and is called the *large topos of* X. If it is unacceptable to enlarge $\mathscr{U}$, a cardinal c may be chosen, an upper bound of the cardinals of X and of all the topological spaces that will enter in to the reasoning (most often, $Max(card\ X,\ card\ R)$ will suffice!), and $\mathbf{Sp}/X$ is then replaced by the subcategory $\mathbf{Sp}'/X$ formed by those X' over X such $card\ X' \leq c$, provided with the
317 induced topology. The topos of sheaves on this site is denoted $TOP(X)$. For clarity of exposition, let us suppose that the first definition has been adopted.

The advantage of the large topos of X over the small is that the site which defines it contains $\mathbf{Sp}/X$ as a full subcategory; as the topology of this site is clearly less fine than the canonical topology, it is seen that the canonical functor from $\mathbf{Sp}/X$ to $TOP(X)$, associating with every space X' on X the sheaf that it represents, is *fully faithful*. Consequently, *a space X' over X is known up to*

[6] The introduction of these sites and toposes is due to Giraud, who has also highlighted their advantages over the traditional “small” site.

X-isomorphism when the sheaf $(\in\ TOP(X))$ *that it defines is known*; thus the notion of sheaf on (the large site of) X can be considered as a *generalisation* of that of the topological space above X, by which all the constructions of sheaf theory have meaning for topological spaces over X.

Thus, when G is a group object of the category $\mathbf{Sp}/X$ of topological spaces over X, the classifying topos B_G (2.4) can be associated with it, classified by the cohomology groups, classified by the homotopy groups etc. (defined as the corresponding invariants of the $\mathscr{V}$-topos B_G). In particular, when X is a point space, G is identified with an ordinary topological group. It can be verified, given 'standard' local conditions which ensure that the singular cohomology of the cartesian products G^n coincides with the sheaf theoretic cohomology (for constant coefficients, let us say), for example if G is locally contractable, that the cohomology of the classifying topos of G is canonically isomorphic to that of the classifying space of G in the usual topological sense.

318 The introduction of classifying toposes (via "large sites") has the advantage over classifying spaces of providing a richer theory, notably because it provides useful cohomological invariants for coefficients not necessarily constant or locally constant. Furthermore, the definition considered here is clearly adaptable to other familiar contexts: differentiable manifolds, manifolds or analytical spaces (whether real or complex), schemes. Notably, this point of view allows the link to be made between the study of characteristic classes from the traditional point of view and the "arithmetical" point of view, by considering the "classical groups" as arising from those schemes defined on the ring of integers; cf. [7] for pointers in this direction. Furthermore, the general results of J. Giraud [3] on the classification of group extensions, developed in the very general and supple framework of toposes, can, thanks to "large toposes", specialise to results on the classification of extension of topological groups, or of real or complex Lie groups, results which seemed to be hardly known to topologists except in the case of extensions to the Abelian kernel [11].

2.6 Topos of form $\hat{\mathcal{C}}$

Let $\mathcal{C}$ be a small category. Thus the category $\hat{\mathcal{C}}$ of the $\mathscr{U}$-presheaves on $\mathcal{C}$ is obviously a $\mathscr{U}$-topos, because it is of the form $\tilde{\mathcal{C}}$, where $\mathcal{C}$ is given the indiscrete topology. Below, we will give some details on the relationship between $\mathcal{C}$ and $\hat{\mathcal{C}}$. Here we simply note that a topos $\mathcal{E}$ equivalent to a topos of the form $\hat{\mathcal{C}}$ is of a fairly unusual nature, since *it has a small generating family* (X_i) *composed of connected*

projective objects, i.e. of objects X such that the functor $Y \mapsto \mathbf{Hom}(X, Y)$
319 transforms epimorphisms into epimorphisms and sums into sums: it is sufficient, in fact, to take in $\hat{\mathcal{C}}$ the generating family made up of functors represented by the $X \in Ob(\mathcal{C})$. Furthermore we note that if in a topos $\mathcal{E}$ there is a covering family $X_i \longrightarrow X$, with X_i that are projective and connected, then every other covering family of X is bounded above (I 4.3.2, 4.3.3) by the preceding. Consequently, in a topos $\mathcal{E}$ of the form $\hat{\mathcal{C}}$ every object X has a covering family that is an upper bound to all others. A topos of the form $Top(X)$ (2.1), where X is a topological space whose points are closed, only has the preceding property when X is discrete.

When the category $\mathcal{C}$ has a single object, $\mathcal{C}$ is identified with a monoid G. A presheaf on $\mathcal{C}$ is thus identified with a set on which G acts on the *right* (because it is a functor $G^\circ \longrightarrow \mathbf{Set}$), and $\hat{\mathcal{C}}$ is the topos of right monoid actions, which can also be written B_{G°, taking into account 2.3: it is the *topos of monoids of* G° *actions (the opposite monoid to G)*. When G is a group, using the isomorphism $g \mapsto g^{-1}$ of G to G°, the classifying topos B_G of 2.3 is recovered.

2.7 Classifying topos of a progroup

2.7.1 Let $\mathcal{G} = (G_i)_{i \in I}$ be a projective system of groups, with $G_i, I \in \mathscr{U}$. The projective system is assumed to be *strict*, i.e. the transition morphisms $G_j \longrightarrow G_i$ are surjective. If E is a set, the following structure is called an *action of* $\mathcal{G}$ *on* E (say, to the left): a) a family $(E_i)_{i \in I}$ of subsets of E, with union E; b) for each $i \in I$, an action of the group G_i on the set E_i; this data is further assumed to be
320 subject to the following condition: for $j \geq i$, E_i is the closed subset of E_j made up of the elements fixed under the kernel group of $G_j \longrightarrow G_i$. It is also said that $\mathcal{G}$ acts to the (left) on E if a (left) action of $\mathcal{G}$ on E is given. The sets $\in \mathscr{U}$ provided with a $\mathcal{G}$ action, obviously form a category. It is immediately observed, thanks to Giraud's criteria, that this category is a $\mathscr{U}$-topos. It is denoted $B_{\mathcal{G}}$ and called the *classifying topos of* $\mathcal{G}$. When I has an initial object i_0, setting $G = G_{i_0}$, the classifying topos of 2.3 is again recovered.

2.7.2 Another important example is that where the groups G_i are finite, such that

$$G = \varprojlim G_i$$

is a totally discontinuous compact topological group, or *profinite group*. An operation of $\mathcal{G}$ on E thus reverts to an operation of G on E that is continuous, or amounting to the same thing, such that the stabilizer of each point of E is an open subgroup of G. The classifying topos $B_{\mathcal{G}}$ will also be written B_G, where, of course, G must be considered as having the profinite topology.

2.7.3 It is easy to check, using the comments in 2.6, that the topos $B_{\mathcal{G}}$ defined by a strict projective system $\mathcal{G} = (G_i)_{i \in I}$ of groups is equivalent to a topos of the form $\hat{\mathcal{C}}$ only if this projective system is essentially constant; in the case of a profinite group, this means that the group is in fact finite.

2.7.4 The following geometric interpretation of the classifying topos B_G of a discrete group G is useful as it provides the correct geometric intuition for these toposes. (Cf. also, in the same way, 4.5, 5.8, 5.9 and 7.2 below.) Let X be
321 a connected topological space, locally connected and locally simply connected, x a point of X, G its fundamental group at x. (N.B. it is known that, up to isomorphism, every discrete group G can be thus obtained.) Thus the Galois theory of coverings of X supplies an equivalence between the category B_G of G-sets and the category of the *étale coverings* of X, i.e. spaces X' over X that are locally X-isomorphic to the X-spaces of the form $X \times I$, where I is a discrete space. (cf SGA1 V 4,5). Furthermore, this last can be interpreted as the category of locally constant sheaves on X, i.e. the category of locally constant objects (IX 2.0) of the topos $Top(X)$.

When G is a profinite group, there is an analogous geometric interpretation of B_G, as the category of X-schemes which are finite étale coverings of a connected scheme X, provided with a geometric point x and an isomorphism $G \cong \pi_1(X, x)$. It is also known that any profinite group can be obtained as a fundamental group of a suitable connected scheme (a field spectrum so to speak). Finally, progroups are also encountered (not necessarily profinite nor essentially constant) in the classification of coverings of connected and locally connected spaces which are not locally simply connected, and in the classification of étale coverings, not necessarily finite or ind-finite, of non-normal connected schemes. For this last case, cf. SGA 3 X 6.

Exercise 2.7.5 Define for a topos $\mathcal{E}$ the notion of connectedness, of local connectedness [7], of simple connectedness and of local simple-connectedness. Define the notion of the constant and locally constant object of $\mathcal{E}$ (cf. IX 2.0). Let
322 $f : \mathcal{E}' \longrightarrow \mathcal{E}$ be a topos morphism (3.1), with $\mathcal{E}'$ simply connected (for ex-

[7] cf. 8.7 1).

ample $\mathcal{E}'$ the point topos (2.2)), and $\mathcal{E}$ connected and locally connected. Define a strict progroup $\pi_1(\mathcal{E}, f) = (G_i)_{i\in I} = \mathcal{G}$ (called the *fundamental progroup of $\mathcal{E}$ over f*) and an equivalence of categories between B_G and the category of locally constant objects of $\mathcal{E}$ [8]. Show that when $\mathcal{E}$ is locally simply connected, $\pi_1(\mathcal{E}, f)$ is essentially constant and is identified with an ordinary discrete group $\pi_1(\mathcal{E}, f)$, which is called the *fundamental group of $\mathcal{E}$ over f*. Show that each strict progroup $\mathcal{G}$ is isomorphic (as a progroup) to the fundamental progroup of a suitable connected and locally connected topos over a suitable f, with $\mathcal{E}'$ the point topos (take $\mathcal{E} = B_{\mathcal{G}}$, and $f : \mathcal{E}' \longrightarrow \mathcal{E}$ defined by the forgetful functor $f^* : \mathcal{E} \longrightarrow \mathcal{E}' = \mathbf{Set}$). When $\mathcal{G}$ is essentially constant, i.e. isomorphic (as a progroup) to an ordinary discrete group, prove that $\mathcal{E}$ above can be taken as locally simply connected (again taking $\mathcal{E} = B_G$).

2.8 Example of a false topos

Let $\mathcal{G} = (G_i)_{i\in I}$ be a strict progroup, where I is a filtered order and where $i > j$ implies that $G_i \longrightarrow G_j$ is not an isomorphism. Let us assume that $\boldsymbol{card}(I) \notin \mathscr{U}$. Let us consider the category of sets $E \in \mathscr{U}$ on which $\mathcal{G}$ acts to the left (2.7.1). It is a $\mathscr{U}$-category, and it is clear as in 2.7.1 that this category satisfies the conditions a), b), c) of 1.1.2. However it is not a $\mathscr{U}$-topos, because it is clear that it does not have a generating family which is $\mathscr{U}$-small. It is also clear that it is not a $\mathscr{V}$-topos for any universe $\mathscr{V}$.

3 Topos morphisms

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Definition 3.1 *Let $\mathcal{E}$ and $\mathcal{E}'$ be two $\mathscr{U}$-toposes. A* morphism[9] *from $\mathcal{E}$ to $\mathcal{E}'$ (or sometimes, by abuse of language, a* continuous map *from $\mathcal{E}$ to $\mathcal{E}'$) is a triple $u = (u_*, u^*, \varphi)$ consisting of functors*

$$u_* : \mathcal{E} \longrightarrow \mathcal{E}' \ , \ u^* : \mathcal{E}' \longrightarrow \mathcal{E}$$

and an adjuction isomorphism of bifunctors on $X' \in Ob(\mathcal{E}')$, $\quad Y \in Ob(\mathcal{E})$:

$$\varphi : \mathbf{Hom}_{\mathcal{E}}(u^*(X'), Y) \xrightarrow{\cong} \mathbf{Hom}_{\mathcal{E}'}(X', u_*(Y))$$

[8] SGA 3 X 6 could serve for inspiration.

[9] *Translator*: Now better known as a *geometric morphism*

the functor u^* also being subject to the condition that it is left exact, *i.e. commutes with finite projective limits. The functor u_* is called the* direct image functor *of the topos morphism u, the functor u^* is called the* inverse image functor *of the topos morphism u and the isomorphism φ is called the* adjunction isomorphism *of u.*

3.1.1 From now on, unless stated otherwise, for a topos morphism $u : \mathcal{E} \longrightarrow \mathcal{E}'$, the corresponding direct image and inverse image functors will be called u_* and u^* [10]. It is noted that, u_* being right adjoint to u^* and u^* being left adjoint to u_* by the adjunction isomorphism φ, each of the two functors u_*, u^* determines the other up to unique isomorphism, in accordance with the well known properties of adjoint functors [14]. In practice, on a case by case basis, it can be more convenient to define a topos morphism $u : \mathcal{E} \longrightarrow \mathcal{E}'$, either by giving $u^* : \mathcal{E}' \longrightarrow \mathcal{E}$, or by giving $u_* : \mathcal{E} \longrightarrow \mathcal{E}'$; in the first case, it is simply necessary to check that
324 the given functor u^* has a right adjoint, and that it is left exact. In the second, that the given functor u_* has a left adjoint that is left exact. In either case we obtain from the given part, thanks to the *choice* of an adjoint functor and an adjunction morphism, a topos morphism $u : \mathcal{E} \longrightarrow \mathcal{E}'$, and this last will be "unique up to unique isomorphism" in terms of the given u^* resp. u_*, in a reasonably clear sense, which will, furthermore, be made entirely clear below (3.2.1).

3.1.2 If $u : \mathcal{E} \longrightarrow \mathcal{E}'$ is a topos morphism, it follows from the properties of adjunct functors (I 2.11) that the direct image functor $u_* : \mathcal{E} \longrightarrow \mathcal{E}'$ commutes with projective limits, and the functor $u^* : \mathcal{E}' \longrightarrow \mathcal{E}$ commutes with inductive limits[11]. As it is also assumed that this last is left exact, i.e. commutes with finite projective limits, it is seen in particular that u^* is *exact*. Further *it is the inverse image functor* u^*, in the ordered pair (u_*, u^*), *which has the most remarkable exactness properties*. These properties ensure that for every type of algebraic structure Σ whose data can be described in terms of "arrow data" between base sets and sets deduced from these by repeated application of finite projective limit and any inductive limit operations, and for any "object of $\mathcal{E}'$ with a Σ-structure" (a notion which has meaning thanks to the internal exactness properties of the topos $\mathcal{E}'$ (II 4.1)), its image in u^* has the same structures. Rather than go into the rather tedious task of giving a precise meaning to this statement and justifying it in a for-

[10] Sometimes u^* may be written, u^{-1}, cf.   .

[11] Moreover, (1.5 and 1.8), for a given functor $u_* : \mathcal{E} \longrightarrow \mathcal{E}'$ resp. $u^* : \mathcal{E}' \longrightarrow \mathcal{E}$, this functor has a left adjoint (resp. right) if and only if it commutes with projective $\mathscr{U}$-limits (resp. with inductive $\mathscr{U}$-limits).

325 mal manner, we recommend that the reader clarifies it and convinces himself of its validity for types of structures such as: group, ring, module over a ring, comodule over a ring, bialgebra over a ring or torsor under a group. (In these examples, the first three types of structure are defined exclusively in terms of finite projective limits, whilst the others implicitly require constructions also calling on inductive limits.) Furthermore, the functor u^* *"commutes"* with all the usual functorial operations on terms of such structures and specifically with all the operations that can be expressed in terms of finite $\varinjlim$ and $\varprojlim$: free object constructions (e.g. free groups or modules) generated by an object, tensor products (cf. Section 12 below) etc.

As for the direct image functor $u_* : \mathcal{E} \longrightarrow \mathcal{E}'$, which commutes with projective limits, it therefore "respects" every algebraic structure on an object (or a family of objects) of $\mathcal{E}$, definable exclusively in terms of projective limits (such as the group, ring or module over a ring structures of the preceding examples). On the other hand, the functor u_* is not, in general, right exact, i.e. it does not, in general, commute with finite inductive limits, and does not even, in general, transform epimorphisms into epimorphisms (it is, moreover, this lack of exactness of the functor u_* that is the source of its cohomological properties, which will be studied (from the point of view of commutative homological algebra) in the following lecture). Consequently, it does not extend, in general, to a functor on
326 objects of type comodule, bi-algebra or torsor under a group, and does not, in general, commute with operations such as "free module generated by", tensor product of modules etc.

3.1.3 In practice, when there is a functor $f : \mathcal{E} \longrightarrow F$ from one $\mathscr{U}$-topos to another, it is worth being explicit about all its exactness properties, including the possible existence of left or right adjoint functors (cf. footnote [8?] - **problem**!!), in order to understand the "geometric nature" of f, an understanding that will generally be an indispensable guide to the correct geometric intuition of the situation. Thus, if it is established that f commutes with all inductive limits and with finite projective limits, f can be written as

$$f = u^*,$$

where

$$u : \mathcal{F} \longrightarrow \mathcal{E}$$

is a topos morphism, i.e. it is possible to interpret f as the "inverse image" functor of a "continuous map" between toposes. When f commutes with all projective limits, and so has a left adjoint, and if this last (which *a priori* commutes with all inductive limits) *also* commutes with finite projective limits, then f can be written as

$$f = v_*,$$

where

$$v : \mathcal{E} \longrightarrow F$$

327 is a topos morphism. In certain circumstances, it can happen that f satisfies both of the properties considered (cf. 4.10 for an example). In this case, it is possible to introduce simultaneously the two topos morphisms

$$u : \mathcal{F} \longrightarrow \mathcal{E} \quad \text{and} \quad v : \mathcal{E} \longrightarrow \mathcal{F},$$

which give rise to a set of three adjoint functors (I 5.3):

$$v^* \dashv \text{[12]}\, v_* = u^* = f \dashv u_* \quad .$$

It is important to carefully distinguish the two topos morphisms u and v, or one risks losing the geometric intuition of the situation.

Note in connection with this that if between two toposes $\mathcal{E}$, $\mathcal{F}$ there is a series of three adjoint functors

$$e \dashv f \dashv g \quad (e, g : \mathcal{F} \rightrightarrows \mathcal{E},\ f : \mathcal{E} \to \mathcal{F}),$$

such that f commutes with inductive limits and with all projective limits, then it can always be written in the form u^*, where $u : \mathcal{F} \longrightarrow \mathcal{E}$ is a topos morphism. So, g is written $g = u_*$. Of course, e, on the other hand, can only be written in the form v^* (and thus f in the form v_*) if it also commutes with finite projective limits. This will obviously be the case if it is itself the right adjoint of a fourth functor d. Without a condition of this nature,

$$e = u_!$$

is often written for the left adjoint of an inverse image functor $f = u^*$, when such a left adjoint exists, this notation being in keeping with the example of an open immersion $u : X \longrightarrow Y$ of topological spaces. Likewise, if e is of the form v^*, i.e. f of the form v_*, the right adjoint of a direct image functor v_* is sometimes written $g = v^!$, when this functor exists [13].

328 **3.2** Let $\mathcal{E}$, $\mathcal{E}'$ be two $\mathscr{U}$-toposes, and

$$u \;=\; (u_*, u^*, \varphi)\;,\; v \;=\; (v_*, v^*, \psi) \;:\; \mathcal{E} \rightrightarrows \mathcal{E}'$$

two topos morphisms from $\mathcal{E}$ to $\mathcal{E}'$. Every morphism from u_* to v_* (in the sense of the category $\mathbf{Hom}(\mathcal{E}, \mathcal{E}')$ of functors from $\mathcal{E}$ to $\mathcal{E}'$) is called a *morphism from u to v*. The morphisms of topos morphisms compose in the obvious manner, and in this way a category is defined which is in fact a *$\mathscr{U}$-category* (I 7.8) denoted

$$\mathbf{Homtop}(\mathcal{E}, \mathcal{E}'),$$

and called the *category of morphisms* (or *of continuous maps*) *from $\mathcal{E}$ to $\mathcal{E}'$*. Thus a functor

$$\mathbf{Homtop}(\mathcal{E}, \mathcal{E}') \;\longrightarrow\; \mathbf{Hom}(\mathcal{E}, \mathcal{E}')$$

$$u \;\mapsto\; u_*$$

is defined in the obvious way, a functor which is fully faithful by the definition of morphisms in the domain (but is not injective on objects).

3.2.1 It should be noted that, if u and v are given as above, the theory of adjoint functors supplies a canonical bijection

$$\mathbf{Hom}(u_*, v_*) \xrightarrow{\sim} \mathbf{Hom}(v^*, u^*);$$

in particular, a *contravariant functor* on $\mathbf{Homtop}(\mathcal{E}, \mathcal{E}')$ is obtained:

$$\mathbf{Homtop}(\mathcal{E}, \mathcal{E}')^\circ \;\longrightarrow\; \mathbf{Hom}(\mathcal{E}, \mathcal{E}')$$

$$u \;\mapsto\; u^*.$$

Following geometric intuition, by which the direct image functor u_* "goes in the same direction" as the continuous map that gives rise to it, the direction of the arrows for the morphisms between topos morphisms can be defined *in terms of the direct image functors*, and not in terms of the inverse image functors (even though it is these last that, we have seen, possess the characteristic exactness properties
329 of the notion of topos morphism).

[12]*Translator*: The current standard notation $\dashv$ is adopted here, though not used in the original

[13]Cf. below for the case of Abelian sheaves.

3.2.2 Having defined the category $\mathbf{Homtop}(\mathcal{E}, \mathcal{E}')$ of morphisms from the topos $\mathcal{E}$ to the topos $\mathcal{E}'$, the notion of *isomorphism* between the two morphisms u, v from $\mathcal{E}$ to $\mathcal{E}'$ is also defined. In practice, there is no need to make a fundamental distinction between two isomorphic topos morphisms, at least not when there is a *canonical* isomorphism between the two (just as it is often unnecessary to distinguish between two objects of a category when a canonical isomorphism between them can be obtained). Let us point out concerning this that most often, when dealing with topos morphism diagrams and questions of the commutativity of such diagrams (a notion that has meaning thanks to (3.3)), we are only concerned with commutativity up to ("canonical") isomorphism; by abuse of language, these diagrams are effectively treated as commutative diagrams.

One can try to justify this abuse of language by introducing the set $\mathbf{Homtop}(\mathcal{E}, \mathcal{E}')/\cong$ of morphisms from $\mathcal{E}$ to $\mathcal{E}'$ up to isomorphism, and by calling any such isomorphism class a morphism rather than following definition 3.1. But this runs into very serious difficulties which present themselves every time an attempt is made to identify two isomorphic objects of a category without having a canonical isomorphism between them. Experience shows that such a point of view is impracticable and that it is necessary to retain the "narrow" notion 3.2 of morphism between topos morphisms, even if it means being obliged, sometimes, to battle with compatibilities between canonical isomorphisms [14].

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3.3.1 Let $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{E}''$ be three $\mathscr{U}$-toposes, and consider the topos morphisms

$$u : \mathcal{E} \longrightarrow \mathcal{E}' , \; v : \mathcal{E}' \longrightarrow \mathcal{E}''.$$

The theory of adjoint functors gives us an adjunction isomorphism between the composite functors $v_* u_*$ and $u^* v^*$, in terms of the adjunction isomorphisms for the couples (u_*, u^*) and (v_*, v^*). Also, the functor $u^* v^*$ is left exact, as it is the composition of two left exact functors. Consequently, a morphism from $\mathcal{E}$ to $\mathcal{E}''$ is found, which is called the *composition of morphisms u and v*, and is written:

$$vu : \mathcal{E} \longrightarrow \mathcal{E}''.$$

So, it can be trivially verified that the composition of morphisms is associative, and that for every $\mathscr{U}$-topos $\mathcal{E}$, there is a morphism of $\mathcal{E}$ to itself that is a left and right unit for the composition: it is the morphism $(id_{\mathcal{E}}, id_{\mathcal{E}}, \varphi)$, where φ is the

[14] For examples of such battles (victorious it would seem) we refer the reader to Hakim's book on relative schemes [9].

obvious adjunction isomorphism of $id_{\mathcal{E}}$ with itself. So, let $\mathscr{V}$ be a universe such that $\mathscr{U} \in \mathscr{V}$. A category is defined

$$\mathscr{V} - \mathscr{U} - \mathbf{Top},$$

where the objects are $\mathscr{U}$-toposes that are $\in \mathscr{V}$, the arrows are morphisms between such $\mathscr{U}$-toposes, and the composition of the arrows is as has just been explained.

3.3.2 In fact, the composition map

$$Homtop(\mathcal{E},\mathcal{E}') \times Homtop(\mathcal{E}',\mathcal{E}'') \longrightarrow Homtop(\mathcal{E},\mathcal{E}'')$$

is the object part of a "composition functor of morphisms":

$$\mathbf{Homtop}(\mathcal{E},\mathcal{E}') \times \mathbf{Homtop}(\mathcal{E}',\mathcal{E}'') \longrightarrow \mathbf{Homtop}(\mathcal{E},\mathcal{E}''),$$

331 whose effect on morphisms is the usual "convolution product" operation for morphisms between functors (here the direct image functors). These composition functors satisfy a (strict) associativity property, for four toposes $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{E}''$, $\mathcal{E}'''$, specifying the associativity of composition of topos morphisms. It can also be said, in language that is starting to become familiar [2] [9], that the $\mathscr{U}$-toposes are objects (or 0-arrows) of a 2-category, whose 1-arrows are topos morphisms, and whose 2-arrows are the morphisms of topos morphisms.

It is the fact that the $\mathscr{U}$-toposes (elements of a universe $\mathscr{V}$) form a 2-category, and no longer just an ordinary category like the ordinary topological spaces, that constitutes from a technical point of view the most important difference between the theory of toposes and that of topological spaces. This fact is the source of certain technical complications which have already been referred to, but it is also, when compared to traditional topology, the source of fundamentally new facts.

3.4 The fact that the $\mathscr{U}$-toposes (elements of a universe $\mathscr{V}$) form a 2-category (3.3.2) allows, in particular, the definition of the notion of *equivalence of two $\mathscr{U}$-toposes $\mathcal{E}$, $\mathcal{E}'$*: we say that $\mathcal{E}$ and $\mathcal{E}'$ are *equivalent* if topos morphisms $u : \mathcal{E} \longrightarrow \mathcal{E}'$ and $v : \mathcal{E}' \longrightarrow \mathcal{E}$ exist, such that the composites vu and uv are isomorphic to the identity morphisms on $\mathcal{E}$ and $\mathcal{E}'$ respectively; and say that
332 the morphisms u and v are *quasi-inverse equivalences* of one another.

It is immediate that for the topos morphism $u : \mathcal{E} \longrightarrow \mathcal{E}'$ to be an equivalence, it is necessary and sufficient that u_* is an equivalence, or amounting to the same thing, that u^* is an equivalence. (Use the fact that a functor $f : \mathcal{E} \longrightarrow \mathcal{E}'$ between two toposes which is an equivalence is at the same time of the form u_* and of the form v^*, with u and v topos morphisms); and for $\mathcal{E}$ and $\mathcal{E}'$ to be equivalent in the sense of the preceding paragraph, it is necessary and sufficient that they are equivalent as categories (i.e. as objects of the 2-category $\mathscr{V}$-$\mathbf{Cat}$). As usual, this shows that the notion of equivalence just introduced does not depend on the choice of universe $\mathscr{V}$, made in 3.3.1.

3.4.1 Practically, it is not often necessary to fundamentally distinguish between equivalent $\mathscr{U}$-toposes, just as it is not often necessary to distinguish between two equivalent categories, provided however that there exists an explicit equivalence from one to the other, or at least an equivalence defined up to unique isomorphism. Here the notion of topos equivalence replaces the traditional notion of homeomorphism between two topological spaces. See the example 4.2 below for the precise relationship between these two notions.

4 Examples of topos morphisms

Here we will go over the examples of Section 2 again, using the notion of topos morphism. The comments in 2.0 are equally applicable to this section. Generally, the universe $\mathscr{U}$ is implicit.

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4.1 The topos $Top(X)$ for variable topological space X

4.1.1 A continuous map

$$f : X \longrightarrow Y$$

of topological spaces is canonically associated with a topos morphism

$$Top(f) \text{ or } f \ : \ Top(X) \longrightarrow Top(Y),$$

with the notation of 2.1. When $Top(X)$ is defined as $\widetilde{\mathcal{O}(X)}$, the most convenient description of $Top(f)$ is by the sheaf direct image functor

$$f_* \ : \ Top(X) \longrightarrow Top(Y),$$

defined by the formula

$$f_*(F) \;=\; F \circ f^{-1},$$

where

$$f^{-1} \;:\; \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$$

is the obvious functor $U \longmapsto f^{-1}(U)$. It has already been noted that this functor is continuous and left exact, and so the functor f_* above is well defined and has a left adjoint f^* which is left exact (III 1.9.1). Strictly, of course, the topos morphism $Top(f)$ depends on the choice of the left adjoint f^* of f_*, and thus is only defined up to canonical isomorphism. In the following such phenomena will not be especially commented on.

Taking the "étale spaces" point of view with respect to $Top(X)$, the inverse image functor

$$f^* \;:\; Top(Y) \;\longrightarrow\; Top(X)$$

334 is the most convenient for defining the topos morphism $Top(f)$, by simply putting

$$f^*(Y') \;=\; X \times_Y Y'$$

for every étale space Y' over Y; it is obvious that the fibred product is indeed an étale space on X, and that the functor f^* thus obtained is left exact and commutes with any $\varinjlim$, and, therefore, defines a topos morphism $Top(f)$. As for the compatibility of the two definitions obtained, we refer the reader to [8].

When we are given two composable continuous maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

there is a canonical isomorphism

$$Top(gf) \;\cong\; Top(g)Top(f)$$

of topos morphisms. These *transitivity isomorphisms*, for three composable continuous maps f, g, h satisfy a compatibility condition that we will not write out here, and which is none other than that considered in SGA I VI 7.4 B) (for $\mathcal{E} \;=\; \mathbf{Sp}^\circ$). This condition can be expressed by saying that for variable X in the category $\mathbf{Sp}$,

$$X \;\mapsto\; Top(X)$$

defines a "pseudo-functor"

(4.1.1.1) $$\mathscr{U} - \mathbf{Sp} \;\longrightarrow\; \mathscr{V} - \mathscr{U} - \mathbf{Top},$$

or, in the terminology of 2-categories, there is a *non-strict* functor of 2-categories [9]. In practice we will allow ourselves, by abuse of language, to identify $Top(gf)$ with $Top(g)Top(f)$, i.e. to reason as if (4.1.1.1) were a true functor of ordinary
335 categories. An analogous abuse of language will be allowed in the other examples that are dealt with below.

4.1.2 The preceding considerations immediately extend to the case of toposes associated with topological spaces with groups actions (2.3). If $f = (f^{Sp}, f^{Gp})$,

$$f \;:\; (X, G) \;\longrightarrow\; (Y, H)$$

is a morphism of spaces with group actions (where

$$f^{Sp} \;:\; X \;\longrightarrow\; Y \text{ and } f^{Gp} \;:\; G \;\longrightarrow\; H$$

are a continuous map and a group homomorphism respectively, compatible in an obvious way), there is a topos morphism

$$Top(f) \text{ or } f \;:\; Top(X, G) \;\longrightarrow\; Top(Y, H)$$

associated with it, whose definition is left to the reader. When the groups G and H are trivial groups, the definition of 4.1.1 is recovered; when, on the other hand, it is the spaces X and Y that are reduced to a point, the inverse image functor is the "group action restriction functor"

$$f^* \;:\; B_H \;\longrightarrow\; B_G,$$

that associates with each H-set the G-set that it defines using $f : G \longrightarrow H$. This example will be encountered again in other forms in 4.5 and 4.6.1.

4.1.3 Given a continuous map of topological spaces $f : X \longrightarrow Y$, a morphism on the corresponding "large toposes" (2.5) can be associated with it

$$TOP(f) \text{ or } f \;:\; TOP(X) \;\longrightarrow\; TOP(Y),$$

336 defined most conveniently by the inverse image functor

$$f^* \;:\; TOP(Y) \;\longrightarrow\; TOP(X) \quad ,$$

which is none other than the *restriction functor*. This morphism $TOP(f)$ is a special case of the so called "inclusion" morphism for an induced topos, which will be studied in Section 5. Thus it is seen that, with the same reservations as in 4.1.1, the topos $TOP(X)$ can be considered as a functor in X, for variable X in $\mathbf{Sp}$.

4.2 Faithfulness properties of $X \mapsto Top(X)$

We aim to specify to what extent a topological space X can be reconstructed in terms of the topos $Top(X)$, and to this end it is useful to describe, for two spaces X and Y, the category of morphisms from $Top(X)$ to $Top(Y)$ (3.2), so as to be able to specify the faithfulness properties of the "functor" $X \mapsto Top(X)$. We limit ourselves to stating the various results we come across, referring the reader to [9] for the details. The reader who wishes to verify these results for himself could refer to exercise 7.8.

4.2.1 Recall that a topological space X is called *sober* if every irreducible closed subset of X has exactly one generic point. Let us point out that almost all spaces used in practice are sober; it is thus so particularly for a separated space, and more generally for a space for which all points are closed, or for the underlying space of a scheme. If X is a topological space, there is associated with it (*loc. cit.* or
337 EGA 0_{I}, re-issued) a sober topological space X_{sob} and a continuous map

(4.2.1.1) $$\varphi : X \longrightarrow X_{sob}$$

which is *universal* for continuous maps from X to sober spaces; in other words, a left adjoint functor $X \mapsto X_{sob}$ to the inclusion functor $\mathbf{Sob} \longrightarrow \mathbf{Sp}$ of the category of sober spaces into that of "any" topological spaces (the quotes reminding us that there is a universe!) is constructed. Explicitly the construction is to take as points of X_{sob} the irreducible closed subsets of X, and as open sets the sets of the form U', where U is an open set of X and where $U' \subset X_{sob}$ designates the set of irreducible closed subsets of X that intersect with U. The map (4.2.1.1) is obtained by associating with each $x \in X$ the closure of $\{x\}$. The space X is sober if and only if the preceding map is bijective and thus a homeomorphism.

Note that the functor

$$\varphi^{-1} : \mathcal{O}(X_{sob}) \longrightarrow \mathcal{O}(X)$$

induced by φ is an isomorphism, which implies that the topos morphism

$$Top(\varphi) : Top(X) \longrightarrow Top(X_{sob})$$

defined by φ is also an isomorphism. This explains in advance why X_{sob} must necessarily be introduced into the question of reconstructing X from $Top(X)$: as this last depends only on X_{sob} up to isomorphism, the question can only have an affirmative answer if X is sober. We specify below (7.1) how X_{sob} can actually be
338 reconstructed in terms of $Top(X)$, by interpreting its points as the "points" of the topos $Top(X)$ (or as fibre functors).

4.2.2 On every topological space, an order $\leq$ can be introduced for which we have

$$x \leq y \Longleftrightarrow \overline{\{x\}} \subseteq \overline{\{y\}} \text{ i.e. } x \in \overline{\{y\}}$$

(which is also expressed by saying that x is a *specialization* of y, or that y is a *generalisation* of x). For a space of the form X_{sob}, this is none other than the inclusion relationship between irreducible closed subsets of X.

With this in place, the so called "specialization" order can be introduced on the set of maps of a space X into another space Y, derived from that on Y, namely

$$f \leq g \iff f(x) \leq g(x) \quad \text{for every } x \in X.$$

With these conventions, we have the following result:

4.2.3 *Let X, Y be two topological spaces, with Y* sober*, and let f and g be two continuous maps from X to Y. Then there is at most one morphism from $Top(f)$ to $Top(g)$, and for there to be one, it is necessary and sufficient that f specializes g. Finally, every topos morphism $Top(X) \longrightarrow Top(Y)$ is isomorphic to a morphism of the form $Top(f)$, where $f : X \longrightarrow Y$ is a continuous map*
339 *(uniquely determined thanks to the first assertion).*

If the category $cat(I)$ associated with an ordered set I is defined by declaring that for $i \geq j$ there is exactly one arrow from j to i, the preceding result can be summarised by stating that there is a canonical equivalence of categories

$$cat(\mathbf{Hom_{Sp}}(X, Y)) \xrightarrow{\simeq} \mathbf{Homtop}(Top(X), Top(Y))$$

(where the second term has been defined in 3.2).

The formal conclusion of these results is:

Corollary 4.2.4 *a) Let $f : X \longrightarrow Y$ be a continuous map. Then $Top(f) : Top(X) \longrightarrow Top(Y)$ is a topos equivalence if and only if $f_{sob} : X_{sob} \longrightarrow Y_{sob}$ is a homeomorphism (therefore, when X and Y are sober, if and only if f is a homeomorphism).*

b) Let X and Y be topological spaces. For $Top(X)$ and $Top(Y)$ to be equivalent, it is necessary and sufficient that X_{sob} and Y_{sob} are homeomorphic (thus, if X and Y are sober, it is necessary and sufficient that X and Y are homeomorphic).

4.3 Morphisms to the final topos: constant objects, section functors

Let us designate by $\mathcal{P}$ the standard final topos, i.e. $\mathcal{P} = \mathbf{Set}$ (2.2). Let $\mathcal{E}$ be any topos, we now show that *up to unique isomorphism, there exists a unique topos morphism*

(4.3.0.1) $$f : \mathcal{E} \longrightarrow \mathcal{P}$$

more precisely, that *the category* $\mathbf{Homtop}(\mathcal{E}, \mathcal{P})$ *is equivalent to the point cat-*
340 *egory*: for two objects of this category, there thus exists a unique arrow from one to the other, and it is an isomorphism. (This justifies to a certain extent the nomenclature "final topos".) For this, let us recall (3.2.1) that $\mathbf{Homtop}(\mathcal{E}, \mathcal{P})$ is equivalent to the full subcategory of $\mathbf{Hom}(\mathcal{P}, \mathcal{E})^\circ$ consisting of those functors

$$f^* : \mathcal{P} = \mathbf{Set} \longrightarrow \mathcal{E}$$

that commute with $\varinjlim$ and are left exact. Let e be a point set, thus every set X is written canonically as the "sum of X copies of e", from which it follows that the category of functors $g : \mathbf{Set} \longrightarrow \mathcal{E}$ that commute with $\varinjlim$ is equivalent to the category $\mathcal{E}$, by associating to each object g the object $g(e)$ of $\mathcal{E}$. So, g is reconstructed in terms of $T = g(e)$, up to unique isomorphism, by $g(I) = T \times I$, setting $T \times I = \coprod_{i \in I} T_i$, with $T_i = T$ for all $i \in I$. That the functor in I thus defined by T indeed commutes with $\varinjlim$ follows from the fact that it is obviously right adjoint to the functor $X \mapsto \mathbf{Hom}(T, X)$ from $\mathcal{E}$ to $\mathbf{Set}$. That said, for g to be left exact, it is clearly necessary that $g(e) = T$ be a final object of $\mathcal{E}$ (since e is a final object of $\mathbf{Set}$), and it follows easily from the fact that in $\mathcal{E}$ "the sums are universal" (1.1.2 b)) that this condition is also sufficient. Therefore, the category of inverse image functors f^* is equivalent to the category of final objects of $\mathcal{E}$, which is obviously itself equivalent to the final category.

From the above, it is seen that the choice of a morphism (4.3.0.1) is essentially equivalent to that of a final object of $\mathcal{E}$, i.e. $e_{\mathcal{E}}$. In terms of this, we thus have canonical isomorphisms of functors

(4.3.0.2) $$f^*(I) \cong e_{\mathcal{E}} \times I = \text{the sum of } I \text{ copies of } e_{\mathcal{E}} \quad \text{for } I \in Ob(\mathbf{Set}),$$

341 and

(4.3.0.3) $$f_*(X) = \mathbf{Hom}(e_{\mathcal{E}}, X) \quad \text{for } X \in Ob(\mathcal{E}) .$$

4.3.4 The two preceding functors will play an important role in what follows. For a set I, the object $f^*(I) = e_{\mathcal{E}} \times I$ of (4.3.0.2) is called the *constant object* *with value* I in $\mathcal{E}$ (or, when $\mathcal{E}$ is implemented as a category $\widetilde{\mathcal{C}}$ in terms of a site $\mathcal{C}$, *constant sheaf with value* I on $\mathcal{C}$). It will often also be written $I_{\mathcal{E}}$, or $I_{\mathcal{C}}$ when $\mathcal{E}$ is defined by the site $\mathcal{C}$. The fact that $I \mapsto I_{\mathcal{E}}$ is the inverse image functor of a topos morphism specifies its exactness properties, which imply in particular that this functor respects all the usual types of algebraic structures, transforming a group into a group object of $\mathcal{E}$ etc (3.1.2). When there is, for example, a group G of $\mathcal{E}$, it is said to be a *constant group* (or, if need be, a *constant sheaf of groups*) if it is isomorphic to a group of the form $(G_O)_{\mathcal{E}}$, where G_O is an ordinary group. The same terminology holds for every other type of "algebraic" structure, in the sense (more or less) specified in 3.1.2.

4.3.5 It should be noted that the functor $I \mapsto I_{\mathcal{E}}$ is not necessarily fully faithful (nor even faithful: take $\mathcal{E}$ to be the "empty topos" (2.2)), thus a constant object of $\mathcal{E}$ does not, in general, determine up to unique isomorphism the set I that generates it. We say that $\mathcal{E}$ is 0-*acyclic*, or *non-empty connected*, if the functor $I \mapsto I_{\mathcal{E}}$ is fully faithful. An equivalent condition is, from the general properties of adjoint functors, that the adjunction morphism

$$I \longrightarrow f_*(f^*(I)) = f_*(I_{\mathcal{E}}) = \mathbf{Hom}(e_{\mathcal{E}}, I_{\mathcal{E}})$$

342 is an isomorphism (such that the functor (4.3.0.3) allows the recovery of the "value" of a constant object of $\mathcal{E}$). It is easily checked that it is necessary and sufficient for this that $e_{\mathcal{E}}$ is not the initial object $\phi_{\mathcal{E}}$ of $\mathcal{E}$, i.e. that $\mathcal{E}$ is not an "empty topos" (which expresses the *faithfulness* of the functor $I \mapsto I_{\mathcal{E}}$ [15]), and that $e_{\mathcal{E}}$ is a *connected* object of $\mathcal{E}$, i.e. is not the sum of two objects of $\mathcal{E}$ that are not "empty" (i.e. that are not the initial objects of $\mathcal{E}$).

4.3.6 The functor (4.3.0.3) is often also called the *section functor* and written $\Gamma_{\mathcal{E}}$ or $\Gamma(\mathcal{E}, -)$ or simply Γ:

(4.3.6.1) $$\mathbf{Hom}(e_{\mathcal{E}}, X) = \Gamma_{\mathcal{E}}(X) = \Gamma(\mathcal{E}, X) = \Gamma(X).$$

It is a functor which commutes with all projective limits, but is not, in general, right exact, whose derived functors (on Abelian group objects) will be studied in the next lecture.

[15] or again the fact that the functor is conservative (I 6.3).

4.4 Morphisms of the "empty topos"

Let ϕ be an empty topos, which thus corresponds to a category of sheaves Φ equivalent to the final category (2.2). Let $\mathcal{E}$ be a topos. The category of functors from $\mathcal{E}$ to Φ is obviously equivalent to the point category, and every such functor commutes with inductive and projective limits (trivially in fact), and thus can be considered as an inverse image functor f for a topos morphism $\phi \longrightarrow \mathcal{E}$. It follows from this that the *category* $\mathbf{Homtop}(\phi, \mathcal{E})$ *is equivalent to the point category*, and in particular that *there exists up to unique isomorphism one and*
343 *only one topos morphism*

$$\phi \longrightarrow \mathcal{E}. \tag{4.4.0.2}$$

This justifies to a certain extent the terminology "initial topos" introduced in 2.2.

The topos morphism

$$\mathcal{E} \longrightarrow \phi \tag{4.4.0.3}$$

can also be determined; it can quickly be verified that such a morphism exists if and only if the initial object of $\mathcal{E}$ is also a final object, i.e. if and only if $\mathcal{E}$ itself is an "empty topos", and that in this case the category $\mathbf{Homtop}(\mathcal{E}, \phi)$ is again equivalent to the point category. The unique morphism (4.4.0.2) (modulo isomorphism) is thus a topos equivalence (in this case).

4.5 The classifying topos B_G for variable group G

4.5.1 Let $\mathcal{E}$ be a topos, and

$$f : G \longrightarrow H$$

be a group homomorphism in $\mathcal{E}$. From this a "group action restriction" functor

$$f^* : B_H \longrightarrow B_G,$$

can be found, where the notation is that of 2.4. It is trivial that this functor commutes with inductive and projective limits, so *a fortiori* it can be interpreted as the inverse image functor associated with a topos morphism

$$B_f \text{ or } f : B_G \longrightarrow B_H.$$

344 The corresponding direct image functor

$$f_* \ : \ B_G \ \longrightarrow \ B_H$$

can easily be made explicit by the formula

$$f_*(X) \ = \ \mathbf{Hom}_G(H_S, X),$$

where X is an object of X with a left G action, where H_S is H with its left G action induced by f, and where $\mathbf{Hom}_G$ is the subobject of the object $\mathbf{Hom}$ as defined below (10); H acts on $\mathbf{Hom}_G(H_S, X)$ to the left by using the right action of H on H_S by *right* translations.

As the inverse image functor f^* commutes with any $\varprojlim$ (and not only finite $\varprojlim$), it is itself the right adjoint of a functor

$$f_! \ : \ B_G \ \longrightarrow \ B_H,$$

such that there is a set of three adjoint functors as in 3.1.3:

$$f_! \ \dashv \ f^* \ \dashv \ f_*.$$

$f_!$ is easily made explicit by the formula

$$f_!(X) \ = \ H \times^G X,$$

where the second term designates the "contracted product", induced from the action of G on X (left action) and on H (right action via right translation and f). It is defined as the quotient of $H \times X$ under the G action given by

$$g.(h, x) \ = \ (h(fg)^{-1}, gx).$$

The functor $f_!$, being left adjoint, obviously commutes with inductive limits, but it is not, in general, left exact (i.e. it cannot be considered in its turn as an inverse
345 image functor of a topos morphism $B_H \ \longrightarrow \ B_G$). In fact, it is easily verified that it can only be left exact if $f \ : \ G \ \longrightarrow \ H$ is an isomorphism. Likewise, the functor f_*, which commutes with projective limits, is not, in general, right exact, and so, *a fortiori* does not, in general, have a right adjoint. At least when $\mathcal{E}$ is the point topos, i.e. when G and H are ordinary groups, f_* is right exact only if f is an isomorphism.

When there is a second morphism of groups $g : H \ \longrightarrow \ K$, transitivity up to canonical isomorphism can be established as in 4.1.1, such that, with the usual reservations, the classifying topos B_G can be considered to depend *functorially* on the group G. The task of generalising this functorial behaviour to the case where both G *and* the topos $\mathcal{E}$ vary simultaneously is left to the reader.

4.5.2 The topos $B_{\mathcal{G}}$ for a variable progroup $\mathcal{G}$

The task of specifying the covariant character of the topos $B_{\mathcal{G}}$ (2.7) with respect to $\mathcal{G}$ is left to the reader, using as a template the procedure given in 4.5.1. It should be noted, however, that in the case of a morphism $f : \mathcal{G} \longrightarrow \mathcal{H}$ of progroups which are not essentially constant, the corresponding topos morphism $B_{\mathcal{G}} \longrightarrow B_{\mathcal{H}}$ does not, in general, permit the definition of a functor $f_!$ (whose right adjoint is the restriction functor of progroup operators f^*). Supposing, to simplify the statement, that $\mathcal{G}$ and $\mathcal{H}$ are defined by the *profinite* groups G and H, it is easily seen that $f_!$ exists if and only if the image of the morphism considered, $f : G \longrightarrow H$, is of finite index in H, and in this case $f_!$ is given by the same
346 formula as in 4.5.

4.6 The topos $\hat{\mathcal{C}}$ for a variable category $\mathcal{C}$

4.6.1 Let

$$f : \mathcal{C} \longrightarrow \mathcal{C}'$$

be a functor from a category $\mathcal{C} \in \mathscr{U}$ to another $\mathcal{C}'$, therefore there is a functor

$$f^* : \hat{\mathcal{C}}' \longrightarrow \hat{\mathcal{C}}$$

$$f^*(F) = F \circ f.$$

It is obvious that this functor commutes with projective and inductive limits, so, *a fortiori* it can be considered as the inverse image functor of a topos morphism

$$\hat{f} \text{ or } f : \hat{\mathcal{C}} \longrightarrow \hat{\mathcal{C}}'.$$

The corresponding direct image functor

$$f_* : \hat{\mathcal{C}} \longrightarrow \hat{\mathcal{C}}'$$

is none other than the functor also written f_* in I 5.1. Furthermore, (as could be foreseen from the fact that f^* also commutes with any $\varprojlim$) f^* also has a left adjoint

$$f_! : \mathcal{C} \longrightarrow \mathcal{C}',$$

(which was also written $f_!$ in I 5.1). Thus, a set of three adjoint functors

$$f_! \dashv f^* \dashv f_*$$

is immediately obtained, the first, moreover, being an extension of $f : \mathcal{C} \longrightarrow \mathcal{C}'$ to categories of presheaves (via the usual extension of $\mathcal{C}$, $\mathcal{C}'$ to categories $\hat{\mathcal{C}}$, $\hat{\mathcal{C}}'$). It should be noted that the functor $f_!$ is not, in general, left exact, nor f_* right exact, which thus dispels all ambiguity as to the direction of the variance of topos
347 $\hat{\mathcal{C}}$ associated with a variable category $\mathcal{C}$: this topos is a *covariant* functor in $\mathcal{C}$, with the usual reservations about transitivity isomorphisms (cf 4.1.1). When the set of objects of $\mathcal{C}$ and of $\mathcal{C}'$ is reduced to a single element, such that $\mathcal{C}$ and $\mathcal{C}'$ are identified with monoids G, G', the corresponding toposes are the classifying toposes B_{G° and $B_{(G')^\circ}$, and we recover the same variance with respect to G, as in 4.1.2 and 4.5 (where the restriction to groups instead of monoids is not relevant).

4.6.2 The 2-functorial dependence of the topos $\hat{\mathcal{C}}$ with respect to $\mathcal{C}$ can be specified by introducing for two categories $\mathcal{C}, \mathcal{C}' \in \mathscr{U}$ a canonical functor

$$\mathbf{Hom}(\mathcal{C}, \mathcal{C}') \longrightarrow \mathbf{Homtop}(\hat{\mathcal{C}}, \hat{\mathcal{C}}'). \tag{4.6.2.1}$$

This functor still needs to be defined on the arrows, and for this we note that $f \mapsto f^*$ allows the codomain to be identified (up to category equivalence) with a full subcategory of $\mathbf{Hom}(\hat{\mathcal{C}}', \hat{\mathcal{C}})^\circ$ (3.2.1). Now, if $f, g : \mathcal{C} \longrightarrow \mathcal{C}'$ are two functors, every morphism $u : f \longrightarrow g$ of functors defines a morphism of functors $F \circ g \longrightarrow F \circ f$ for each functor in $F \in Ob(\hat{\mathcal{C}}')$ (F being *contravariant*), which thus defines a natural transformation $g^* \longrightarrow f^*$ and consequently defines a morphism $\hat{f} \longrightarrow \hat{g}$ as required.

When $\mathcal{C}$ is the point category, $\hat{\mathcal{C}}$ is the point topos (2.2) denoted $\mathcal{P}$, and (4.6.2.1) becomes a functor

$$\mathcal{C}' \longrightarrow \mathbf{Homtop}(\mathcal{P}, \hat{\mathcal{C}}') \equiv \mathbf{Pt}(\hat{\mathcal{C}}') \tag{4.6.2.2}$$

from $\mathcal{C}'$ to the "category of points" of $\hat{\mathcal{C}}'$, which will be studied in Section 6.
348 This functor is not necessarily an equivalence of categories (7.3.3), so, *a fortiori* (4.6.2.1) is not necessarily an equivalence of categories.

On the other hand, the functor (4.6.2.1) is always *fully faithful*. To see why, let us note that if $f, g : \mathcal{E} \longrightarrow \mathcal{E}'$ are two topos morphisms such that $f_!$ and $g_!$ are defined (3.1.3), it follows from the theory of adjoint functors that there are canonical isomorphisms

$$\mathbf{Hom}(f_!, g_!) \cong \mathbf{Hom}(g^*, f^*) \cong \mathbf{Hom}(f_*, g_*) \equiv \mathbf{Hom}(f, g).$$

Applying this to functors of the form $\hat{f}$, $\hat{g}$ associated with $f, g : \mathcal{C} \longrightarrow \mathcal{C}'$, the promised result is obtained, taking into account that the natural extension map $\mathbf{Hom}(f, g) \longrightarrow \mathbf{Hom}(f_!, g_!)$ is bijective (I 7.8).

4.6.3 The question of when the functor (4.6.2.1) is an equivalence of categories can be asked, i.e. when is it essentially surjective? This is a special case of the question of characterising all topos morphisms $\hat{\mathcal{C}} \longrightarrow \hat{\mathcal{C}}'$. More generally, if $\mathcal{C}$ is a category $\in \mathscr{U}$ and $\mathcal{E}$ is a topos, we can ask for a characterisation of all topos morphisms

$$f : \hat{\mathcal{C}} \longrightarrow \mathcal{E}.$$

The category of these morphisms is equivalent to the opposite of the category of functors $f^* : \mathcal{E} \longrightarrow \hat{\mathcal{C}}$ that commute with all inductive limits and with finite projective limits. Interpreting the functors $\mathcal{E} \longrightarrow \hat{\mathcal{C}}$ as functors $\mathcal{E} \times \mathcal{C}^\circ \longrightarrow \mathscr{U}-\mathbf{Set} = \mathbf{Set}$, or as the functors $F : \mathcal{C}^\circ \longrightarrow \mathbf{Hom}(\mathcal{E}, \mathbf{Set})$, the exactness property considered is expressed by the condition that for every object X of $\mathcal{C}$, the functor $F(X) : \mathcal{E} \longrightarrow \mathbf{Set}$ commutes with any inductive limits and with finite projective limits, (or, as we will say in Section 6, $F(X)$ is a "fibre
349 functor" on $\mathcal{E}$). An equivalent statement is that $F(X)$ is the inverse image functor of a topos morphism $\mathcal{P} \longrightarrow \mathcal{E}$, so that a canonical equivalence of categories is established

$$\mathbf{Homtop}(\hat{\mathcal{C}}, \mathcal{E}) \xrightarrow{\simeq} \mathbf{Hom}(\mathcal{C}, \mathbf{Pt}(\mathcal{E})) , \tag{4.6.3.1}$$

where, to shorten notation, the category of points of the topos $\mathcal{E}$ is designated

$$\mathbf{Pt}(\mathcal{E}) = \mathbf{Homtop}(\mathcal{P}, \mathcal{E}) .$$

When $\mathcal{E}$ is of the form $\hat{\mathcal{C}}'$, it is immediately seen from the definitions that the composite of (4.6.2.1) and the preceding equivalence (4.6.3.1) is the functor

$$\mathbf{Hom}(\mathcal{C}, \mathcal{C}') \longrightarrow \mathbf{Hom}(\mathcal{C}, \mathbf{Pt}(\hat{\mathcal{C}}')) \tag{4.6.3.2}$$

defined by $F \mapsto i \circ F$, where $i : \mathcal{C}' \longrightarrow \mathbf{Pt}(\hat{\mathcal{C}}')$ is the canonical embedding (4.6.2.2). It follows immediately that *in order for (4.6.2.1) to be an equivalence, it is necessary and sufficient that* $\mathcal{C}$ *be empty or that (4.6.2.2) be essentially surjective* (thus an equivalence of categories). This last condition on $\mathcal{C}'$ is satisfied in certain interesting cases, and notably when $\mathcal{C}'$ is the single object category defined by a group G (7.2.5).

Remark 4.6.4 The fact of that a topos $\hat{\mathcal{C}}$ has been associated with an arbitrary category $\mathcal{C}$ suggests that a category $\mathcal{C}$ has invariants of a topological nature (cohomology groups, homotopy groups etc), just like a topos. The cohomology groups

of $\hat{\mathcal{C}}$ with coefficients in an Abelian object group F (in the general sense studied in Lecture V) are none other than the values of the derived functors $\varprojlim^{(n)}$ of
350 the functor $\varprojlim$, already familiar to topologists. J.L. Verdier and (independently) D.G. Quillen have verified that when limited to constant coefficients, or more generally locally constant coefficients, these cohomology groups are identified with the cohomology groups of the semi-simplical set $Nerve(\mathcal{C})$ canonically associated with $\mathcal{C}$ [5, No. 212, prop 4.1] and that, furthermore, up to isomorphism in the "homotopy category" of [2], every semi-simplical set can be obtained using a category $\mathcal{C}$. In return for a suitable notion of homotopy type for toposes, that we will not specify here, it can be said that the semi-simplical homotopy types of topologies are none other than the homotopy types of toposes of the special form $\hat{\mathcal{C}}$, more generally toposes $\mathcal{E}$ where every object has a covering that refines all the others (2.6) [16]. In the absence of this condition on $\mathcal{E}$, the best that can be done is to express its homotopy type by a suitable *projective system* of semi-simplical sets [1].

4.7 The topos $\widetilde{\mathcal{C}}$ for a variable site $\mathcal{C}$ (cocontinuous functors)

Let

$$f : \mathcal{C} \longrightarrow \mathcal{C}'$$

be a cocontinuous functor (III 2.1) between sites $\in \mathscr{U}$, i.e. a functor such that the functor $\hat{f}_*$ or f_*[17] from $\hat{\mathcal{C}}$ to $\hat{\mathcal{C}}'$ (4.6.1) takes $\widetilde{\mathcal{C}}$ to $\widetilde{\mathcal{C}}'$, that is to say induces a functor

(4.7.0.1) $$\widetilde{f}_* : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{C}}'$$

351 that makes the functor diagram

(4.7.0.2) $$\begin{array}{ccc} \widetilde{\mathcal{C}} & \xrightarrow{\widetilde{f}_*} & \widetilde{\mathcal{C}}' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ \hat{\mathcal{C}} & \xrightarrow{\hat{f}_*} & \hat{\mathcal{C}}' \end{array}$$

commute, where i, i' are inclusion functors. It has been seen (III 2.3) that the functor $\widetilde{f}_*$ has a left adjoint $\widetilde{f}^*$, and that this last is left exact. In other words, $\widetilde{f}_*$ is

[16] and, more generally still, toposes which are "locally ∞-connected" in an obvious sense that we will leave to the reader the task of specifying.

[17] *Translator*: The original indicates that f_* is given by the assignment $F \mapsto F \circ f$, however this assignment is left adjoint to f_*, so the reference to it has been dropped.

the direct image functor of a topos morphism

$$\widetilde{f} \text{ or } f : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{C}'} ,$$

the corresponding inverse image functor being, of course, $\widetilde{f}^*$. Taking the left adjoints of the functors into account, the commutative diagram (4.7.0.2) gives, moreover, a commutative diagram up to isomorphism

(4.7.0.3)
$$\begin{array}{ccc} \widetilde{\mathcal{C}} & \xleftarrow{\widetilde{f}^*} & \widetilde{\mathcal{C}'} \\ {\scriptstyle \boldsymbol{a}}\uparrow & & \uparrow{\scriptstyle \boldsymbol{a}'} \\ \hat{\mathcal{C}} & \xleftarrow{\hat{f}^*} & \hat{\mathcal{C}}' \end{array}$$

where $\boldsymbol{a}$ and $\boldsymbol{a}'$ are the "associated sheaf" functors, a diagram that immediately re-establishes the formula (III 2.3)

$$\widetilde{f}^* = \boldsymbol{a}\, \hat{f}^* i'.$$

The transitivity property of topos morphisms $\hat{f} : \hat{\mathcal{C}} \longrightarrow \hat{\mathcal{C}}'$ implies the same property for topos morphisms $\widetilde{f} : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{C}'}$ associated with cocontinuous functors, so it can be said that *the topos $\widetilde{\mathcal{C}}$ varies functorially with* $\mathcal{C}$ in a *covariant*
352 manner, when cocontinuous functors are taken as "morphisms" of sites.

In all the cases encountered up until now, the cocontinuous functor f used is also continuous, that is to say (III 1.1) "extends" to a functor

$$\widetilde{f}_! : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{C}'}$$

commuting with inductive limits, and which is left adjoint to $\widetilde{f}^*$, such that there is a set of three adjoint functors

$$\widetilde{f}_! \dashv \widetilde{f}^* \dashv \widetilde{f}_*.$$

It should be noted that the functor $\widetilde{f}_!$ is not, in general, left exact, nor $\widetilde{f}_*$ right exact, which dispels all ambiguity as to the direction of the variance of the topos $\widetilde{\mathcal{C}}$ for variable $\mathcal{C}$ by functors which are only assumed cocontinuous whether they are cocontinuous, or indeed continuous and cocontinuous.

Remark 4.7.4 Given a topos morphism

$$F : \mathcal{E} \longrightarrow \mathcal{E}',$$

for a functor $F_!$ left adjoint to F^* to exist, it is necessary and sufficient that $\mathcal{E}$ and $\mathcal{E}'$ can be "realised" (up to equivalence) in the form $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{C}'}$, for two sites $\mathcal{C}, \mathcal{C}' \in \mathscr{U}$, and that a *continuous and cocontinuous* functor $f : \mathcal{C} \longrightarrow \mathcal{C}'$ can be found such that F is identified with $\widetilde{f}$. It is obviously sufficient, and for the necessity, it is sufficient to take for $\mathcal{C}$ and $\mathcal{C}'$ small full generating subcategories of $\mathcal{E}$ and $\mathcal{E}'$ respectively, such that

$$F_!(Ob(\mathcal{C})) \subset Ob(\mathcal{C}'),$$

provided with topologies induced by those of $\mathcal{E}$ and $\mathcal{E}'$, and to take f as the functor induced by $F_!$ (cf 1.2.1). Let us remember (1.8) that the existence of $F_!$, also means that F^* commutes with projective limits, or, which here is equivalent since
353 the functor is left exact, that it commutes with all products.

4.7.5 If we ask which topos morphisms $F : \mathcal{E} \longrightarrow \mathcal{E}'$ arise via a cocontinuous (though not necessarily continuous) functor of sites, it is similarly seen that the answer is the following: the set of objects X on $\mathcal{E}$ such that the functor $X' \mapsto \mathbf{Hom}(X, F^*(X'))$ from $\mathcal{E}'$ to $\mathbf{Set}$ commutes with projective limits (or, equivalently, with products), must be a *generating family of* $\mathcal{E}$.

4.8 The topos morphism $\widetilde{\mathcal{C}} \longrightarrow \hat{\mathcal{C}}$ for a site $\mathcal{C}$

Let $\mathcal{C}$ be a small site, with which are therefore associated two toposes $\widetilde{\mathcal{C}}$ and $\hat{\mathcal{C}}$ (the second independent of the topology placed on $\mathcal{C}$). In II 3.4 the "associated sheaf" functor

$$\boldsymbol{a} : \hat{\mathcal{C}} \longrightarrow \widetilde{\mathcal{C}}$$

has been defined, and it has been established that it is left exact and commutes with inductive limits. It is thus the inverse image functor of a topos morphism

$$p : \widetilde{\mathcal{C}} \longrightarrow \hat{\mathcal{C}}\,,\; p^* = \boldsymbol{a},$$

the corresponding direct image functor being the canonical inclusion

$$i = p_* : \widetilde{\mathcal{C}} \longrightarrow \hat{\mathcal{C}}.$$

It is well known that this last functor is not, in general, right exact (its derived functors on the Abelian objects give rise to the cohomology presheaves $\mathscr{H}^n(F)$ of Lecture V 2), and that $\boldsymbol{a}$ does not commute, in general, with arbitrary $\varprojlim$, so this dispels any ambiguity as to the direction of the "natural" topos morphism between $\hat{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$.

354 When there is a cocontinuous functor

$$f : \mathcal{C} \longrightarrow \mathcal{C}'$$

of sites, a diagram of topos morphisms

$$\begin{array}{ccc} \widetilde{\mathcal{C}} & \xrightarrow{\tilde{f}} & \widetilde{\mathcal{C}'} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ \hat{\mathcal{C}} & \xrightarrow{\hat{f}} & \hat{\mathcal{C}'} \end{array}$$

arises which is commutative up to canonical isomorphism: it is, in fact, this that expresses the commutativity of the functor diagram (4.7.0.2). It can thus be said that the topos morphism $p : \widetilde{\mathcal{C}} \longrightarrow \hat{\mathcal{C}}$ is functorial in $\mathcal{C}$, when $\mathcal{C}$ is varied by *cocontinuous* functors between sites.

4.9 The effect of a continuous functor of sites. Morphisms of sites

4.9.1 If

$$f : \mathcal{C} \longrightarrow \mathcal{C}'$$

is a continuous functor from $\mathcal{C}$ to $\mathcal{C}'$, i.e. such that the functor $\hat{f}^* : \hat{\mathcal{C}'} \longrightarrow \hat{\mathcal{C}}$ maps $\widetilde{\mathcal{C}'}$ to $\widetilde{\mathcal{C}}$, and thus induces a functor

$$f_s : \widetilde{\mathcal{C}'} \longrightarrow \widetilde{\mathcal{C}},$$

it has been seen (III 1.2) that this last has a left adjoint

$$f^s : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{C}'},$$

which moreover "extends" f in an obvious way. It should be noted that, in general, f^s is not left exact (even if f is also cocontinuous), nor does f_s commute with inductive limits, such that being given f does not allow, without further assumptions, the description of a topos morphism in either direction between $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{C}'}$. The case where f is cocontinuous, i.e. where f_s commutes with inductive limits and can thus be considered as the inverse image functor of a topos morphism
355 $\widetilde{f} : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{C}'}$, has been examined in 4.7. We are going to examine the case

where the functor f^s is left exact, and thus can be considered as an inverse image functor for a topos morphism *in the opposite direction*:

$$Top(f) = g : \widetilde{\mathcal{C}'} \longrightarrow \widetilde{\mathcal{C}}. \tag{4.9.1.1}$$

It should be noted that *care has been taken not to denote this morphism by the letter $\widetilde{f}$ or f, to avoid confusion with the situation in 4.7, where we are following the general recommendations contained in 3.1.3.* We sometimes say that the functor $f : \mathcal{C} \longrightarrow \mathcal{C}'$ is a *morphism of sites from $\mathcal{C}'$ to $\mathcal{C}$* (note, *not from $\mathcal{C}$ to $\mathcal{C}'$*) if it is continuous and the functor f^s is left exact, in other words if a topos morphism (4.9.1.1) exists such that the corresponding inverse image functor

$$g^* : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{C}'}$$

extends the functor f, i.e. the functor diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{C}' \\ {\scriptstyle \epsilon}\downarrow & & \downarrow{\scriptstyle \epsilon'} \\ \widetilde{\mathcal{C}} & \xrightarrow{g^*} & \widetilde{\mathcal{C}'} \end{array}$$

commutes up to isomorphism, where ϵ, ϵ' are the canonical functors of II 4.4.0.

4.9.2 In practice, one often recognises that a continuous functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is a morphism of sites from $\mathcal{C}'$ to $\mathcal{C}$, by the fact that $\mathcal{C}$ has finite projective limits
356 and f commutes with them (III 1.3.5).

With this condition placed on $\mathcal{C}$ (almost always confirmed in practice), and supposing, furthermore, that the topology of $\mathcal{C}'$ is less fine than the canonical topology (almost always confirmed as well), the preceding sufficient condition (f left exact) for f to be a morphism of sites from $\mathcal{C}'$ to $\mathcal{C}$ is, moreover, also necessary.

4.9.3 In the spirit of what has gone before, if $\mathcal{C}$ and $\mathcal{C}'$ are two $\mathscr{U}$-sites, there are grounds for defining the category of morphisms of sites, $\mathfrak{Morsite}(\mathcal{C}', \mathcal{C})$, from $\mathcal{C}'$ to $\mathcal{C}$, as the full subcategory of the *opposite* category $\mathbf{Hom}(\mathcal{C}, \mathcal{C}')^\circ$ of those functors from $\mathcal{C}$ to $\mathcal{C}'$ that really want to be the morphisms of the sites (from $\mathcal{C}'$ to $\mathcal{C}$). In this way, a canonical functor (defined up to unique isomorphism)

$$\mathfrak{Morsite}(\mathcal{C}', \mathcal{C}) \longrightarrow \mathbf{Homtop}(\widetilde{\mathcal{C}'}, \widetilde{\mathcal{C}})$$

is obtained.

Proposition 4.9.4 *Let $\mathcal{E}$ be a $\mathscr{U}$-topos and $\mathcal{C}$ a $\mathscr{U}$-site, thus the functor $f \mapsto f^*|\mathcal{C} = f^* \circ \epsilon_{\mathcal{C}}$, associating with each topos morphism $f : \mathcal{E} \longrightarrow \widetilde{\mathcal{C}}$ the "restriction" to $\mathcal{C}$ of the associated inverse image functor $f^* : \widetilde{\mathcal{C}} \longrightarrow \mathcal{E}$, induces an equivalence of categories*

$$\mathbf{Homtop}(\mathcal{E}, \widetilde{\mathcal{C}}) \xrightarrow{\simeq} \mathfrak{Morsite}(\mathcal{E}, \mathcal{C}) \quad (\hookrightarrow \mathbf{Hom}(\mathcal{C}, \mathcal{E})^\circ).$$

If $\mathcal{C}$ has finite $\varprojlim$, the corresponding fully faithful functor

$$\mathbf{Homtop}(\mathcal{E}, \widetilde{\mathcal{C}}) \longrightarrow \mathbf{Hom}(\mathcal{C}, \mathcal{E})^\circ$$

has an essential image equal to the set of functors $g : \mathcal{C} \longrightarrow \mathcal{E}$ that are left exact and continuous, *or which are left exact and transform any covering family into a*
357 *covering family.*

The last assertion follows from the first thanks to 4.9.2. On the other hand, we deduce from III 1.2 iv) and from IV 1.2 iii) that the functor $G \longrightarrow G \circ \epsilon$ induces an equivalence between the category of *continuous* functors from $\widetilde{\mathcal{C}}$ to $\mathcal{E}$, and the category of *continuous* functors from $\mathcal{C}$ to $\mathcal{E}$, a quasi-inverse functor being obtained by $g \mapsto g^s$ in the notation of *loc. cit.*. On the other hand, by definition, g is a morphism of sites if and only if g^s is the inverse image functor associated with a topos morphism $\mathcal{E} \longrightarrow \widetilde{\mathcal{C}}$, whence the conclusion, thanks to 3.2.1, the "or" coming from III 1.6.

The important point of 4.9.4 is that (when $\mathcal{C}$ has finite $\varprojlim$) to be given a topos morphism $f : \mathcal{E} \longrightarrow \widetilde{\mathcal{C}}$, or a functor $g : \mathcal{C} \longrightarrow \mathcal{E}$ which is left exact and transforms covering families to covering families "amounts to the same thing".

4.9.5 Using the work of 4.9.1 and 4.9.3, it is seen, as usual, that for a variable $\mathscr{U}$-site, via the notion of morphism of sites and of morphism of morphisms of sites that has just been made clear, the topos $\widetilde{\mathcal{C}}$ depends functorially (or more exactly, 2-functorially) on the site $\mathcal{C}$. It is noted that, thanks to the terminology introduced, $\widetilde{\mathcal{C}}$ depends on the site $\mathcal{C}$ in a *covariant* manner.

4.9.6 It immediately follows that (contrary to what happens for the notion of cocontinuous functor, cf 4.7.4) *every* topos morphism $F : \mathcal{E} \longrightarrow \mathcal{E}'$ can be produced using a morphism of sites $f : \mathcal{C} \longrightarrow \mathcal{C}'$ (i.e. using a continuous functor $f : \mathcal{C}' \longrightarrow \mathcal{C}$ such that...): it is sufficient to choose in $\mathcal{E}$ and $\mathcal{E}'$ small full
358 generating subcategories $\mathcal{C}$ and $\mathcal{C}'$ respectively, provided with topologies induced

by those of $\mathcal{E}$ and $\mathcal{E}'$, such that

$$F^*(Ob(\mathcal{C}')) \subset Ob(\mathcal{C}),$$

and to take for f the functor induced by F^*. This explains why most topos morphisms met in practice are actually defined using morphisms of sites (rather than using cocontinuous functors as in 4.7).

4.10 The relationship between the small and large toposes associated with a topological space X

We will reuse the notation of 2.5. In particular, $\mathscr{V}$ is a universe such that $\mathscr{U} \in \mathscr{V}$. We will stray from the convention of 2.1, by designating by $Top(X)$ the topos of $\mathscr{V}$-sheaves (and not of $\mathscr{U}$-sheaves) over X. Thus, we will reason using a $\mathscr{V}$-topos and not a $\mathscr{U}$-topos. (N.B. it is possible to keep within a $\mathscr{U}$-topos by adopting the appropriate convention for the definition of $TOP(X)$, cf. 2.5., such that it is a $\mathscr{U}$-topos.) Having established this, we will define *two* topos morphisms

(4.10.0.1)
$$\begin{cases} f : Top(X) \longrightarrow TOP(X) \\ g : TOP(X) \longrightarrow Top(X) \quad , \quad gf \cong id_{Top(X)}, \end{cases}$$

giving rise to a set of three adjoint functors

(4.10.0.2)
$$g^* = f_! \dashv g_* = f^* \dashv g^! = f_*,$$

359 the notation being that of 3.1.3. We will define, one after the other, the first two functors of this set, the third being defined as right adjoint to the second.

4.10.2 The functor

$$g_* = f^* : TOP(X) \longrightarrow Top(X)$$

is simply defined as the *restriction functor* of $\mathbf{Sp}/X$ to the site $\mathcal{O}(X)$ of open sets of X, which indeed transforms sheaves into sheaves, as follows immediately from the definition. Using caligraphic letters to designate sheaves on the large site X, for a certain sheaf $\mathcal{F}$, its restriction to the site $\mathcal{O}(X)$ is designated by $\mathcal{F}_X$, whence the functor

$$\begin{array}{rcl} Restr : TOP(X) & \longrightarrow & Top(X) \\ \mathcal{F} & \mapsto & \mathcal{F}_X \, . \end{array}$$

It is obvious that this functor commutes with projective $\mathscr{V}$-limits, because these are calculated argument by argument (the existence of the left adjoint, constructed in 4.10.4 below, could also be invoked). I say that it also commutes with inductive $\mathscr{V}$-limits. To convince ourselves of this, a very useful interpretation of the "large" sheaves over X, i.e. the objects of $TOP(X)$, will be given in terms of ordinary sheaves (N.B. these are $\mathscr{V}$-sheaves) on the topological spaces X' over X.

4.10.3 For a large sheaf $\mathcal{F}$ over X, and for every space X' over X ($X' \in \mathscr{U}$ is understood), the "small" sheaf $\mathcal{F}_{X'}$, the restriction of $\mathcal{F}$ to X', is defined in an obvious way, as in 4.10.2. If

$$u \;:\; X'' \longrightarrow X'$$

360 is a morphism of $\mathbf{Sp}/X$, a morphism $u_*(\mathcal{F}_{X''}) \longrightarrow \mathcal{F}_{X'}$, or equivalently, a "*transition morphism*"

$$\varphi_u \;:\; u_*(\mathcal{F}_{X''}) \longrightarrow \mathcal{F}_{X'} \tag{4.10.3.1}$$

is defined in an obvious manner. These morphisms, for variable u, satisfy an obvious transitivity condition for a composite

$$X''' \xrightarrow{u} X'' \xrightarrow{v} X'$$

of morphisms in $\mathbf{Sp}/X$, which the reader is left the task of making explicit. In this way a natural functor is obtained, which goes from the category $TOP(X)$ of large sheaves over X, to the category of systems

$$(\mathcal{F}_{X'})\;(X' \in Ob(\mathbf{Sp}/X))\;,\;(\varphi_u) \quad (u \in \mathcal{M}(\mathbf{Sp}/X)),$$

satisfying the transitivity condition considered and such that, furthermore, for every morphism $u : X'' \longrightarrow X'$ which is an *open immersion* (or, more generally an étale map), the transition morphism φ_u is an isomorphism. As the functors $u^* : Top(X') \longrightarrow Top(X'')$ used in the description of φ_u commute with inductive $\mathscr{V}$-limits, it follows that in the preceding description of the objects of $TOP(X)$ in terms of "small" sheaves $\mathcal{F}_{X''}$, inductive $\mathscr{V}$-limits are calculated argument by argument, i.e. functors of the form $\mathcal{F} \longmapsto \mathcal{F}_X$ commute with inductive $\mathscr{V}$-limits.

In particular, it is so for the functor $\mathcal{F} \longmapsto \mathcal{F}_X$ considered in 4.10.2. Thus, it indeed has a right adjoint (1.5). A left adjoint (whose existence, moreover, follows *a priori* from 1.8) remains to be constructed, and it remains to be verified
361 that this last is left exact. This completes the definition of topos morphisms stated (4.10.1) in terms of functors (4.10.1.1).

4.10.4 The functor

$$g^* \;=\; f_! \;:\; Top(X) \;\longrightarrow\; TOP(X)$$

is obtained by associating with each small sheaf F over X the system of its inverse images $(F_{X'}),\ X' \in Ob(\mathbf{Sp}/X)$ under the structure map $X' \;\longrightarrow\; X$ with transition morphisms φ_u for $u \;:\; X'' \;\longrightarrow\; X'$ given by the transitivity isomorphisms of the inverse images of sheaves (4.1.1). It is directly seen that a *fully faithful* functor

$$Ext \;:\; Top(X) \;\longrightarrow\; TOP(X),$$

is thus obtained, whose essential image consists of those large sheaves $\mathcal{F}$ over X for which all the transitivity morphisms φ_u of 4.10.3.1 are isomorphisms. We leave the reader the task of defining an adjunction isomorphism between this canonical extension function and the restriction function of 4.10.2, proving that this last is right adjoint to the first. This is obvious in terms of the description 4.10.3 of the category $TOP(X)$.

Remarks 4.10.5 a) The construction 4.10.4 shows at the same time that $g^* \;=\; f_!$ is fully faithful (or, equivalently, that $g_* \;=\; f^*$ is a quotient functor to a category of fractions, or that $g^! \;=\; f_*$ is fully faithful). The large sheaves over X which belong to the essential image of the functor $g^* = f_! \;=\; Ext$ merit the name of large *étale sheaves* over X, since they form a category equivalent to that of the
362 ordinary sheaves over X, or indeed to that of the étale spaces over X. Moreover, it is easily seen that if X' is a topological space above X, then the large sheaf over X that it represents is étale in the preceding sense if and only if X' is an étale space over X.

b) The fact that f_* is fully faithful can also be expressed by noting that the adjunction morphism $f^*f_* \;\longrightarrow\; id_{Top(X)}$ is an isomorphism, i.e. that $g_*f_* \;\cong\; id$, i.e. that $gf \;\cong\; id$. Thus g makes $TOP(X)$ a *topos over* $Top(X)$, admitting a "section" f from $Top(X)$.

c) The fact that the functor $g^* \;=\; f_!$ is fully faithful, exact and commutes with inductive $\mathscr{V}$-limits partially justifies a very useful point of view (due to J. Giraud) according to which it is innocuous, in practically every question of sheaf theory over X, to replace the usual or "small" sheaves with the associated "large" sheaves. It is especially so for questions of cohomology because, as g_* is exact, the functors R^ig_* (Lecture V) are null for $i \;>\; 0$, thus (Lecture V) for every "large" sheaf $\mathcal{F}$ over X, there are canonical isomorphisms

$$H^i(TOP(X), \mathcal{F}) \cong H^i(Top(X), \mathcal{F}_X) \;=\; H^i(X, F).$$

Applying this to a sheaf of the form $Ext(F)$, where F is a small sheaf over X, we conclude that (since $Ext(F)_X \cong F$ canonically) there exists a canonical isomorphism

$$H^i(X, F) = H^i(TOP(X), Ext(F)).$$

Thus, *the cohomological invariants of X, calculated via the small or large topos of X, are essentially identical.* Moreover, the same result is valid in non-commutative cohomology.

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Exercise 4.10.6 Let S be a $\mathscr{U}$-site whose topology is less fine than the canonical topology, $\mathcal{M}_0$ a subset of $\mathcal{M}(S)$ satisfying the following conditions:

a) The morphisms of $\mathcal{M}_0$ are quarrable (I 10.3), and $\mathcal{M}_0$ is stable under change of base.

b) $\mathcal{M}_0$ contains all identity arrows and is stable under composition.

c) Any arrow $u : X \longrightarrow Y$ such that there exists a covering family $Y_i \longrightarrow Y$, with each $X \times_Y Y_i \longrightarrow Y_i$ in $\mathcal{M}_0$, is itself a member of $\mathcal{M}_0$.

d) For every $X \in Ob(S)$, every covering family of X is refined by a covering family $(f_i : X_i \longrightarrow X)$, with each $f_i \in \mathcal{M}_0$.

For every $X \in Ob(S)$, let us consider the site $S(X)$ ("small site of X") whose underlying category is the full subcategory of S/X made up of objects whose structural morphism is in $\mathcal{M}_0$, provided with the topology induced (III 3.1) by that of S.

1) For every arrow $u : X \longrightarrow Y$ of S, show that change of base by u from $S(Y)$ to $S(X)$ is continuous and hence induces a functor that commutes with inductive limits (III)

$$S(u)^s : \widetilde{S(Y)} \longrightarrow \widetilde{S(X)}.$$

2) Define an equivalence between the topos $\widetilde{S}$ and the category of systems

$$(F_X)\ (X \in Ob(S))\ ,\ (\varphi_u)\ (u \in \mathcal{M}(S))$$

364 made up of objects $F_X \in Ob(\widetilde{S(X)})$ and, for every arrow $u : X \longrightarrow Y$ in S, a morphism $\varphi_u : S(u)^s(F_Y) \longrightarrow F_X$, these systems being subject to a transitivity condition for any composite $v \circ u$ of morphisms of S, and to the condition that $u \in \mathcal{M}_0$ implies that φ_u is an isomorphism.

3) Define the "restriction" and "extension" functors

$$Res_X \ : \ \widetilde{S/X} \ \longrightarrow \ \widetilde{S(X)}\ , \ Ext_X \ : \ \widetilde{S(X)} \ \longrightarrow \ \widetilde{S/X}.$$

Show that Res_X commutes with small inductive and projective limits and that Ext_X is *fully faithful*, its essential image being made up of sheaves F such that φ_u is an isomorphism for every morphism u of S/X.

4) Define an adjunction morphism making Res_X right adjoint to Ext_X. Conclude that a topos morphism

$$f \ : \ \widetilde{S(X)} \ \longrightarrow \ \widetilde{S/X}$$

exists, making $\widetilde{S(X)}$ a subtopos of $\widetilde{S/X}$ and such that

$$f_! \ = \ Ext_X$$
$$f^* \ = \ Res_X.$$

Show that Ext_X transforms Abelian sheaves to Abelian sheaves.

5) Show that if $S(X)$ has fibred products, Ext_X is exact. Deduce from this that a topos morphism $g \ : \ \widetilde{S/X} \ \longrightarrow \ \widetilde{S(X)}$ exists which is a left retraction of f,
365 i.e. such that

$$g^* \ = \ Ext_X$$
$$g_* \ = \ Res_X.$$

(For an example where $S(X)$ does not have fibred products, take S to be the category of schemes with the étale topology, $\mathcal{M}_0$ to be smooth morphisms and X a Noetherian scheme of dimension > 0.)

6) Show that for every Abelian sheaf F of $\widetilde{S/X}$ there is an isomorphism

$$H^q(X, F) \ \cong \ H^q(X, Res_X F) \quad \forall\, q.$$

Show, using for example hypercoverings, that for every Abelian sheaf G of $\widetilde{S(X)}$ there is an isomorphism

$$H^q(X, G) \ \cong \ H^q(X, Ext_X G) \quad \forall\, q.$$

7) Buy the editor[18] a chocolate medal.

[18] *Translator*: Or the translator!

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