

# On the Parallel between the Suplattice and Preframe approaches to Locale Theory

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## Abstract

This paper uses the locale theory approach to topology. Two descriptions are given of all locale limits, the first description using suplattice constructions and the second preframe constructions. The symmetries between these two approaches to locale theory are explored. Given an informal assumption that open locale maps are parallel to proper maps (an assumption hinted at by the underlying finitary symmetry of the lattice theory but not formally proved) we argue that various pairs of locale theory results are ‘parallel’, that is, identical in structure but prove facts about proper maps on one side of the pair and about open maps on the other. The pairs of results are: pullback stability of proper/open maps, regularity of the category of compact Hausdorff/discrete locales, and theorems on information systems. Some remarks are included on a possible formalization of this parallel as a duality.

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## 1 Introduction

Say we are given two topological spaces  $X$  and  $Y$  and are required to describe the set of opens of the product space  $X \times Y$ . The well known answer is to look at the following subsets of  $X \times Y$ :

$$U \times V = \{(u, v) | u \in U, v \in V\}$$

where  $U, V$  are arbitrary opens of  $X, Y$  respectively. The collection of all such sets, i.e.

$$\beta \equiv \{U \times V | U, V \text{ open in } X, Y\}$$

is closed under finite intersections since  $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$ . Therefore  $\beta$  forms a basis for a topology and the whole topology is formed by taking all unions of sets of the form  $U \times V$ . Equivalently we can note that the topology is formed by taking the least subsuplattice of  $P(X \times Y)$  generated by  $\beta$ . Recall that a suplattice is a poset with arbitrary joins and so the union operation shows that  $P(A)$  is a suplattice for any set  $A$ ; see [JoyTie 84] for background on suplattices.

There is, however, a parallel solution to this problem. Look at the following subsets of  $X \times Y$ :

$$U \odot V \equiv \{(u, v) | u \in U \text{ or } v \in V\}$$

where again  $U, V$  are open subsets of  $X, Y$ . It is easy to check that  $(U_1 \odot V_1) \cup (U_2 \odot V_2) = (U_1 \cup U_2) \odot (V_1 \cup V_2)$ , and so we conclude that the collection

$$\gamma \equiv \{U \odot V | U, V \text{ open in } X, Y\}$$

is closed under finite unions. Therefore to generate a topology from  $\gamma$  it needs to be closed with respect to directed unions and finite intersections. Define  $\tau$  to be the collection of all directed unions of finite intersections of elements of  $\gamma$ . It can be seen that  $\tau$  is closed under directed unions and finite intersections, i.e. it is a subpreframe of  $P(X \times Y)$ . A preframe is a poset with finite meets and directed joins such that finite meets distribute over directed joins, see [JohVic 91]. Clearly  $\tau$  is the least subpreframe of  $P(X \times Y)$  containing  $\gamma$  and finally, by distributivity of  $P(X \times Y)$ ,  $\tau$  is closed under finite unions. So  $\tau$  forms a topology.

We have now defined two topologies for  $X \times Y$ , one is the least subsuplattice of  $P(X \times Y)$  containing all the sets  $U \times V$  for  $U, V$  open in  $X, Y$ , and the other is the least subpreframe of  $P(X \times Y)$  containing all the sets of the form  $U \odot V$  for  $U, V$  open in  $X, Y$ . But

$$\begin{aligned} U \odot V &= (U \times Y) \cup (X \times U) \\ U \times V &= (U \odot \phi) \cap (\phi \odot V) \end{aligned}$$

where  $\phi$  is the empty set and from this a short proof shows that the two topologies are the same. We could have used either approach in order to define the product topology.

The point of this paper is to take this observation to heart, and to ask how much more topology can be expressed in both ways? The first set of results below shows that provided we use the locale theory approach to topology, all limits can be expressed using either suplattice constructions or preframe constructions.

The core of the paper then applies this approach to a discussion of proper and open maps in locale theory. It is shown how to use preframe constructions to discuss proper maps and how to use identical suplattice constructions to discuss open maps. Therefore if we take the preframe approach to locales, the theory of proper maps emerges and, identically but with reversed finitary data,

if we take the suplattice approach to locales the theory of open maps emerges. In this way we argue, informally, that proper maps are parallel to open maps: they are two sides of the same theory.

In detail a number of known locale theory results are examined side by side. For each pair the suplattice view is used to prove one side and the preframe view to prove the other, showing that the proofs have essentially the form. The results discussed are standard ones about proper and open maps: (i) they are pullback stable and the proper/open surjections are regular epimorphisms, (ii) they can be used to define the compact Hausdorff/discrete locales respectively, and the classes of these locales both form regular categories and (iii) both proper and open information system theory can be developed (i.e. based on Scott's information systems, [Scott 82]).

The results are all known (based on original work in [JoyTie 84], [Vermeulen 93], [Vickers 97] and [Townsend 96]), the novelty is in the presentation. The proofs provided here demonstrate in some detail the strong parallel that exists between the theories of the two classes of maps that emerge (open and proper). This parallel was known to Vermeulen in [Vermeulen 93] and the categorical abstraction of each side of the parallel is examined in [Vickers 95]. The details of how the lattice theoretic techniques are parallel is made clear in [Townsend 96] and [Vickers 97]. Johnstone's recent exposition, contained in C1.1, C3.1 and C3.2 of [Johnstone 02], also uses this parallel covering a proportion of the results offered here. Our description goes beyond [Johnstone 02] in its description of the regularity of compact Hausdorff locales and results in information system theory. This paper is a collection of known results presented in a uniform framework. The uniform framework provided, it is hoped, sheds some light on how the techniques of suplattice and preframe theory can be exploited in locale theory.

Finally comments have been included in the section "Further Work" which indicate how it may be possible to formalize this parallel as a duality.

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## 2 Locales

We take the localic approach to topological space theory. The notation is **Loc** for the category of locales. For any locale  $X$  use  $\Omega X$  for the corresponding frame of opens. The category of locales is, by definition, the opposite of the category of frames (denoted **Frm**). A frame is a distributive lattice which is also a preframe. Equivalently it is a complete lattice such that for any subset  $T$  and any element  $a$  the infinite distributivity law

$$a \wedge \bigvee T = \bigvee \{a \wedge t \mid t \in T\}$$

holds. Equivalently, again, a frame is a complete Heyting algebra. For example the poset of opens of any topological space forms a frame. If  $f : X \rightarrow Y$  is a map between locales then  $\Omega f : \Omega Y \rightarrow \Omega X$  is notation for the corresponding frame homomorphism (preserves arbitrary joins and finite meets), i.e. use  $\Omega f$  for  $f^{op}$ . Note that for any continuous map  $f : X \rightarrow Y$  between topological spaces the inverse image function  $f^{-1}$  is a frame homomorphism. The categories of preframes and suplattices are denoted **PFrm** and **Sup** respectively. Preframe homomorphisms preserve finite meets and directed joins, suplattice homomorphisms preserve arbitrary joins.

Whilst **Loc** is not **Top** (the category of topological spaces) a number of observations can help convince us that we are still doing topology when studying locales. One observation is that under a mild separation axiom (sobriety) topological spaces embed in **Loc**. For example compact Hausdorff spaces and discrete spaces are sober and so we can discuss them inside the category of locales. A number of well known topological notions have suitable localic analogies, for example there is a localic Stone-Ćech compactification functor. [Johnstone 82] is the standard reference for further details on how the category of locales provides a context for topological space theory. The localic context is “choice free”, i.e. not dependent on the axiom of choice and so is a logically more general account of topology. Further, broadly speaking at least, the excluded middle is not needed for locale theory and so the localic context is constructive; that is, all the results can be carried out in an arbitrary elementary topos. This paper will focus on proper and open maps in locale theory. Under mild separation axioms, to be outlined below, such maps are in 1-1 correspondence with proper and open maps familiar from topology space theory.

The terminal locale (1) has frame of opens  $\Omega$ . Recall that  $\Omega$  is the set of subsets of the singleton set (i.e.  $\Omega \equiv P\{*\}$ ) and for any locale  $X$ ,  $\Omega! : \Omega \rightarrow \Omega X$  is given by

$$i \mapsto \bigvee^{\uparrow} \{0_{\Omega X}\} \cup \{1_{\Omega X} \mid 1 \leq i\}.$$

where the uparrow ( $\uparrow$ ) indicates that the join is of a directed set. It is also worth recalling that for any  $i, j \in \Omega$  to prove that  $i \leq j$  it is sufficient to show that  $i = 1_{\Omega}$  implies  $j = 1_{\Omega}$ ; this is immediate since  $\Omega \equiv P\{*\}$ .

Locally the example just given in the Introduction is saying:

**Theorem 1** *If  $X, Y$  are two locales then*

- (i)  $\Omega(X \times Y) \cong \Omega X \otimes_{PFrm} \Omega Y$ <sup>1</sup>
- (ii)  $\Omega(X \times Y) \cong \Omega X \otimes_{Sup} \Omega Y$

*Note:* In [JohVic 91] it is shown that preframe presentations present. I.e. given any poset of generators together with relations between preframe terms of generators (i.e. terms constructed from finite meets and directed joins) then the generators map, universally, to a preframe satisfying the relations. In this way the preframe tensor is well defined via its usual presentation by generators and

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<sup>1</sup>See the paragraph *Notes on Results* below for citation details covering the main results.

relations. It is known (e.g. [JoyTie 84]) that suplattice presentations present and so suplattice tensor is also well defined.

**Proof.** (i) Firstly  $\Omega X \otimes_{PFrm} \Omega Y$  is a frame. The map  $\Omega X \times \Omega Y \times \Omega X \times \Omega Y \rightarrow \Omega X \otimes_{PFrm} \Omega Y$  given by

$$(a_1, b_1, a_2, b_2) \longmapsto (a_1 \vee a_2) \odot (b_1 \vee b_2)$$

is a preframe homomorphism in each of its four components. It therefore corresponds to a preframe bihomomorphism

$$\Omega X \otimes_{PFrm} \Omega Y \times \Omega X \otimes_{PFrm} \Omega Y \rightarrow \Omega X \otimes_{PFrm} \Omega Y$$

which can be verified to be binary join since every element of  $\Omega X \otimes_{PFrm} \Omega Y$  is of the form  $\bigvee_j^\uparrow \wedge_i a_{ij} \odot b_{ij}$ . To prove this bihomomorphism is join first verify that it is idempotent. Since join is defined as a preframe bihomomorphism finite meets distribute over finite joins;  $\Omega X \otimes_{PFrm} \Omega Y$  is therefore a distributive lattice (the bottom element is given by  $0_{\Omega X} \odot 0_{\Omega Y}$ ). Therefore  $\Omega X \otimes_{PFrm} \Omega Y$  is a frame. To check that it is the coproduct of the frames  $\Omega X, \Omega Y$  say we are given frame homomorphisms  $\Omega p_1 : \Omega X \rightarrow \Omega Z$  and  $\Omega p_2 : \Omega Y \rightarrow \Omega Z$  for some locale  $Z$  then define  $\Omega q : \Omega X \otimes_{PFrm} \Omega Y \rightarrow \Omega Z$  by  $q(a \odot b) = \Omega p_1(a) \vee \Omega p_2(b)$ .

(ii) Entirely similar argument using the universal characterization of the suplattice tensor. E.g. for the ‘‘frame coproduct’’ part, define  $\Omega q : \Omega X \otimes_{Sup} \Omega Y \rightarrow \Omega Z$  by  $\Omega q(a \otimes b) = \Omega p_1(a) \wedge \Omega p_2(b)$ . ■

### 3 Locale Equalizers and Pullbacks

Having constructed locale products via preframe and suplattice constructions, to complete a discussion of finite locale limits a description of locale equalizers is needed. Before this fact is stated and proved it should be commented that the categories **PFrm** and **Sup** are symmetric monoidal closed. Certainly a tensor for each has been introduced and, further,

**Lemma 2** (i) For any preframe  $A$ ,  $A \otimes_{PFrm} (-) \dashv [A \rightarrow \_]$ ,

(ii) for any suplattice  $A$ ,  $A \otimes_{Sup} (-) \dashv [A \rightarrow \_]$ .

(And the right hand function space functors are well defined.)

**Proof.** ((i) and (ii) together.) Using pointwise join/meet constructions it is a straightforward calculation to prove that the preframe and suplattice function spaces are well defined. Given these calculations the adjunctions are immediate from the fact that the category of sets is cartesian closed. ■

It can be verified that  $\Omega$  is the unit for both the preframe and the suplattice tensor (note that  $\Omega$  is both the free preframe and the free suplattice on the singleton set  $\{*\}$ ). So this completes an outline proof of the fact that **PFrm** and **Sup** are symmetric monoidal closed.

**Theorem 3** If  $f, g : X \rightrightarrows Y$  is a pair of locale maps then the locale equalizer,  $E$ , is define by

(i)  $\Omega E \equiv \mathbf{PFrm}\langle \Omega X \text{ qua preframe} \mid \Omega f(b) \vee a = \Omega g(b) \vee a, \forall b \in \Omega Y, \forall a \in \Omega X \rangle$  and

(ii)  $\Omega E \equiv \mathbf{Sup}\langle \Omega X \text{ qua suplattice} \mid \Omega f(b) \wedge a = \Omega g(b) \wedge a, \forall b \in \Omega Y, \forall a \in \Omega X \rangle$ .

**Notation 4** The notation  $\mathbf{PFrm}\langle G \text{ qua } R_0 \mid R \rangle$  means the free preframe universally generated by  $G$  subject to the relations  $R_0$  and  $R$ . For the expression “ $\Omega X$  qua preframe” take  $R_0$  to be all equations of the form

$$\begin{aligned} (1) &= 1 \\ (a \wedge b) &= (a) \wedge (b) \\ (\vee^\uparrow T) &= \bigvee^\uparrow \{(t) \mid t \in T\} \end{aligned}$$

where the second equation is over all pairs  $a, b$  and the third over all directed subsets  $T$ . Here we are using  $(-)$  to denote the universal map. Thus ‘ $\Omega X$  qua preframe’ is saying, ‘keeping the preframe structure in  $\Omega X$ ’. For example the functor

$$\begin{aligned} F &: \mathbf{PFrm} \rightarrow \mathbf{Frm} \\ A &\mapsto \mathbf{Frm}\langle A \text{ qua preframe} \mid \phi \rangle \end{aligned}$$

is left adjoint to the forgetful functor  $U : \mathbf{Frm} \rightarrow \mathbf{PFrm}$ .

Exactly similarly for suplattices.

**Proof.** (i) The proof is similar to the proof that preframe tensor is the same as locale product construction. Firstly we need to verify that  $A \equiv \mathbf{PFrm}\langle \Omega X \text{ qua preframe} \mid \Omega f(b) \vee a = \Omega g(b) \vee a, \forall b \in \Omega Y, \forall a \in \Omega X \rangle$  is a frame. For any  $a_1 \in \Omega X$  define the preframe map  $\alpha^{a_1} : A \rightarrow A$  by  $\alpha^{a_1}(a_2) = a_1 \vee a_2$  for every  $a_2 \in \Omega X$ . But the assignment  $a_1 \mapsto \alpha^{a_1}$  satisfies the equations “qua preframe” and “ $\Omega f(b) \vee a = \Omega g(b) \vee a$ ”. We have therefore defined a preframe bihomomorphism  $A \times A \rightarrow A$  which can readily be verified to be the join operation. To verify this first check that it is idempotent.

(ii) Identical techniques. ■

**Corollary 5** *If*

$$\begin{array}{ccc} W & \xrightarrow{p_2} & Y \\ p_1 \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

*is a pullback diagram in  $\mathbf{Loc}$  then*

$$\Omega W \cong \mathbf{PFrm}\langle \Omega X \otimes_{\mathbf{PFrm}} \Omega Y \text{ qua preframe} \mid (\Omega f(c) \vee a) \odot b = a \odot (\Omega g(c) \vee b) \forall a \in \Omega X, b \in \Omega Y, c \in \Omega Z \rangle,$$

*and*

$$\Omega W \cong \mathbf{Sup}\langle \Omega X \otimes_{\mathbf{Sup}} \Omega Y \text{ qua suplattice} \mid (\Omega f(c) \wedge a) \otimes b = a \otimes (\Omega g(c) \wedge b) \forall a \in \Omega X, b \in \Omega Y, c \in \Omega Z \rangle.$$

## 4 Arbitrary Locale Limits

To cover all locale limit constructions we must address the issue of constructing filtered colimits of frames.

**Proposition 6** *The forgetful functors*

(i)  $U : \mathbf{Frm} \rightarrow \mathbf{PFrm}$  and

(ii)  $U : \mathbf{Frm} \rightarrow \mathbf{Sup}$

both create filtered colimits.

**Proof.** ((i) and (ii) together.) Recall that, via presentations and relations, colimits in  $\mathbf{PFrm}$  and  $\mathbf{Sup}$  can always be constructed. Given a filtered diagram of frames,  $D : J \rightarrow \mathbf{Frm}$ , consider  $\text{colim}_i UD(i)$  the preframe or suplattice colimit. Then

$$\begin{aligned} \text{colim}_i UD(i) \otimes \text{colim}_j UD(j) &\cong \text{colim}_i (UD(i) \otimes \text{colim}_j UD(j)) \\ &\cong \text{colim}_i (\text{colim}_j (UD(i) \otimes UD(j))) \\ &\cong \text{colim}_{(i,j)} UD(i) \otimes UD(j) \\ &\cong \text{colim}_i (UD(i) \otimes UD(i)). \end{aligned}$$

The first two equivalences are because  $\mathbf{PFrm}/\mathbf{Sup}$  are monoidal *closed* and so the tensor commutes with all colimits. The last equivalence is because  $J$  is filtered. The join/meet operation  $*_{D(i)}$  on the  $D(i)$ s induce a function

$$*_D : \text{colim}_i UD(i) \otimes \text{colim}_i UD(i) \rightarrow \text{colim}_i UD(i)$$

and it may be verified that this is idempotent and then that  $*_D$  is join/meet as needed. The preframe/suplattice  $\text{colim}_i UD(i)$  is then a frame and is the frame filtered colimit. ■

There are therefore two descriptions available for all locale limits. In the next section we begin to apply these descriptions to some separation axioms on locales.

## 5 Compact and Open Locales

A locale  $X$  is *compact* if and only if for every directed subset  $T \subseteq^\uparrow \Omega X$  if  $1_{\Omega X} \leq \bigvee^\uparrow T$  then there exists  $t \in T$  such that  $1_{\Omega X} \leq t$ . For example the opens of any compact topological space is the frame of opens of a compact locale.

Equivalently,

**Theorem 7** *A locale  $X$  is compact if and only if the right adjoint to  $\Omega!$  is a preframe homomorphism.*

**Proof.** This right adjoint (given by  $a \mapsto \bigcup \{1_\Omega \mid 1_{\Omega X} \leq a\}$ ) always exists since frame homomorphisms preserve arbitrary joins. The right adjoint always preserves finite meets. It clearly preserves directed joins iff the locale is compact. (Recall that trivially  $\Omega$  is compact since  $\Omega \equiv P\{*\}$ .) ■

The definition of an open locale is,

**Definition 8** *A locale  $X$  is open if and only if  $\Omega!$  has a left adjoint.*

In contrast to the compactness separation axiom, the left adjoint does not always exist. But if it does exist then it is always a suplattice homomorphism. Also, in contrast, classically all locales are open, whereas not all locales are compact.

**Theorem 9** *Assuming the excluded middle any locale  $X$  is open.*

**Proof.** Define the left adjoint by sending any  $a$  to 1 if  $a$  is not equal to  $0_{\Omega X}$  and to 0 otherwise. ■

This provides examples of open locales. Constructive examples also exist, take  $\Omega X = PA$ , the power set of any set  $A$ . Then the left adjoint to  $\Omega!$  can be defined without use of the excluded middle: send a subset  $A_0 \subseteq A$  to the truth value

$$\bigvee \{1_{\Omega} \mid \exists x \in A_0\}.$$

Note that we have switched from compact to open by replacing right adjoint with left adjoint and preframe with suplattice. It is going to be argued that compact is ‘parallel’ to open and so this last Theorem is significant as it shows that classically one side of the parallel is partially invisible (since, assuming the excluded middle, all locales are open). Classically therefore the theorem “the product of two open locales is open” is not very meaningful. Constructively this is a result that requires proof; its proper parallel is the well known finite Tychonoff theorem.

**Proposition 10** (i) *(finite Tychonoff) If  $X, Y$  are compact then so is  $X \times Y$ ,*  
(ii) *if  $X, Y$  are open then so is  $X \times Y$ .*

**Proof.** (i) The map

$$\Omega \xrightarrow{\Omega!^X} \Omega X \xrightarrow{\cong} \Omega X \otimes \Omega \xrightarrow{Id \otimes \Omega!^Y} \Omega X \otimes \Omega Y$$

is a frame homomorphism and so is  $\Omega!^{X \times Y} : \Omega \rightarrow \Omega X \otimes_{PFrame} \Omega Y \cong \Omega(X \times Y)$  since  $\Omega$  is initial. But this map clearly has a preframe right adjoint if  $\Omega!^X$  and  $\Omega!^Y$  both have preframe right adjoints.

(ii) Identical suplattice argument with left adjoint in place of right adjoint.

■

The general Tychonoff theorem follows from the previous result:

**Theorem 11** (i) *The arbitrary product of compact locales is compact.*

(ii) *The arbitrary product of open locales is open.*

**Proof.** (i) Given compact  $X_i, i \in I$ , it must be verified that filtered colimit  $\text{colim}_{I_0 \subseteq \text{fin } I}^{\text{Frm}} \Omega X_i$  is the frame of opens of a compact locale. Note that transition maps are of the form  $\Omega\pi$  for finitary projection maps  $\pi$  (e.g.

$\pi_1 : X_{i_1} \times X_{i_2} \rightarrow X_{i_1}$ , so  $\Omega\pi_1(a) = a \odot 0$ . But then  $\Omega\pi$  can be written, for example, as

$$\Omega X_{i_1} \xrightarrow{\cong} \Omega X_{i_1} \otimes \Omega \xrightarrow{Id \otimes \Omega!^{X_{i_2}}} \Omega X_{i_1} \otimes \Omega X_{i_2}$$

and so the transition maps have preframe right adjoints which therefore commute with all the right adjoints  $\otimes_{I_0} \Omega X_i \rightarrow \Omega$  (which exist by finite Tychonoff). Hence these right adjoints must extend to a preframe homomorphism  $colim_{I_0 \subseteq^{Frm} I} \otimes_{I_0} \Omega X_i \rightarrow \Omega$  since the filtered colimit is created in **PFrm**. This preframe homomorphism can be seen to be right adjoint to  $\Omega! : \Omega \rightarrow colim_{I_0 \subseteq^{Frm} I} \otimes_{I_0} \Omega X_i$ . To verify this note that the bijections set up by the definition of preframe colimit are always order isomorphisms.

(ii) Identical suplattice argument. ■

## 6 Proper and Open Maps

We have therefore argued informally that compactness is parallel to openness for locales, since they can be defined identically, but by interchanging right adjoint with left adjoint and preframe with suplattice. Continuing this informal approach we have a pair of parallel definitions. They are the same definition but with dual finitary lattice data and with suplattice homomorphisms exchanged with preframe homomorphisms.

**Definition 12** (a)  $f : X \rightarrow Y$  in **Loc** is open if

(i) there exists  $\exists_f : \Omega X \rightarrow \Omega Y$  a suplattice homomorphism left adjoint to  $\Omega f : \Omega Y \rightarrow \Omega X$  and

(ii)  $\exists_f(a \wedge \Omega f(b)) = b \wedge \exists_f(a)$ , for all  $a \in \Omega X, b \in \Omega Y$  (Frobenius),  
and

(b)  $f : X \rightarrow Y$  in **Loc** is proper if

(i) there exists  $\forall_f : \Omega X \rightarrow \Omega Y$  a preframe homomorphism right adjoint to  $\Omega f : \Omega Y \rightarrow \Omega X$  and

(ii)  $\forall_f(a \vee \Omega f(b)) = b \vee \forall_f(a)$ , for all  $a \in \Omega X, b \in \Omega Y$  (coFrobenius).

From the observation that these two definitions are parallel we argue that a whole series of locale theory results are parallel. As an immediate illustrative example the following Proposition allows us to conclude informally that open locales are parallel to compact locales.

**Proposition 13** (i)  $X$  is open iff  $! : X \rightarrow 1$  is open,

(ii)  $X$  is compact iff  $! : X \rightarrow 1$  is proper.

**Proof.** It must be checked that the Frobenius and coFrobenius conditions (part (ii)) of the definitions of open and proper are always true for the maps  $! : X \rightarrow 1$ . This is a straightforward verification given that  $\Omega!(i) = \bigvee^1 \{0_{\Omega X}\} \cup \{1_{\Omega X} \mid 1 \leq i\}$ . ■

The words proper and open are imported from point set topology, so some comment is needed that these localic versions correspond with the usual definitions, at least under certain separation axioms. One way round is easy:

**Lemma 14** *If  $f : X \rightarrow Y$  is a continuous map between topological spaces then*

(a) *if  $f$  is open as a topological map then  $f^{-1} : Opens(Y) \rightarrow Opens(X)$  is the frame homomorphism of an open locale map,*

(b) *if  $f$  is proper as a topological map then, assuming the excluded middle,  $f^{-1} : Opens(Y) \rightarrow Opens(X)$  is the frame homomorphism of a proper locale map*

**Proof.** (a) If  $f$  is open then the direct image function provides a left adjoint to  $f^{-1}$ . Checking the Frobenius condition is routine.

(b) Say  $f$  is proper (i.e. direct image preserves closed subspaces, and fibres are compact). Recall that the right adjoint to  $f^{-1}$  is  $\forall_f$ , given by

$$\forall_f(U) = \bigcup \{V \mid f^{-1}V \subseteq U\}.$$

Now, certainly for any directed collection of opens  $(U_i)_{i \in I}$  in  $X$  we have that  $\bigcup_{i \in I}^\uparrow \forall_f(U_i) \subseteq \forall_f(\bigcup_{i \in I}^\uparrow U_i)$ . To prove the reverse inclusion, say  $x \in \forall_f(\bigcup_{i \in I}^\uparrow U_i)$ . Then  $x \in V$  for some open  $V$  of  $Y$  with  $f^{-1}V \subseteq \bigcup_{i \in I}^\uparrow U_i$  so by compactness of  $f^{-1}\{x\}$  there exists  $i \in I$  such that  $f^{-1}\{x\} \subseteq U_i$ . But, since the direct image of closed subsets is closed,  $\forall_f$  can alternatively be given by

$$\forall_f(U) = [f_\#(U^c)]^c$$

where  $(-)^c$  is set theoretic complement and  $f_\#$  is direct image. It follows that  $x \in \forall_f(U_i)$  since if  $x \in f_\#(U_i^c)$  there would be an element in  $U_i \cap U_i^c$ . Verifying the coFrobenius condition is routine given  $\forall_f(U) = [f_\#(U^c)]^c$ . For example say  $x \in \forall_f(U \cup f^{-1}V)$ , and say, for contradiction that  $x \notin \forall_f(U) \cup V$ . Then  $x \notin V$  and  $x = f(y)$  for some  $y \notin U$ . But then  $x \in f_\#([U \cup f^{-1}V]^c)$  a contradiction. ■

The lemma does have a converse provided  $Y$  is a  $T_D$  space, that is provided every point  $\{y\}$  is an open subspace of its closure (i.e.  $\{y\} = cl\{y\} \cap V$  for some open  $V$ ). For example, Hausdorff. Details of the converse can be found in, for example, [Plewe 97].

## 7 Closed and Open Sublocales

Recall that a locale map  $i : X_0 \hookrightarrow X$  is a sublocale map iff it is a regular monomorphism iff  $\Omega i$  is a surjection.  $i$  is closed iff  $\Omega i$  is of the form  $b \mapsto b \vee a$  for some fixed  $a \in \Omega X$  ( $\Omega X_0 = \uparrow a$ ) and  $i$  is open iff  $\Omega i$  is of the form  $b \mapsto b \wedge a$  for some fixed  $a \in \Omega X$  ( $\Omega X_0 = \downarrow a$ ). See Chapter II, 2.4 of [Johnstone 82] for background on how these correspond to closed/open subspaces. Proper and open maps generalize closed and open sublocales respectively.

**Proposition 15** *Given a sublocale  $i : X_0 \hookrightarrow X$*

- (i)  *$i$  is closed iff  $i$  is a proper map and*
- (ii)  *$i$  is open iff  $i$  is an open map.*

**Proof.** (i) If  $i$  is a closed sublocale then the inclusion of  $\Omega X_0$  in  $\Omega X$  is a preframe homomorphism right adjoint to  $\Omega i$  and the coFrobenius condition is satisfied. Conversely, if  $i$  is proper as a map and  $\Omega i$  is a surjection then  $\Omega X_0 \cong \uparrow \forall_i(0_{\Omega X_0})$ .

(ii) If  $i$  is an open sublocale then the inclusion of  $\Omega X_0$  in  $\Omega X$  is a suplattice homomorphism left adjoint to  $\Omega i$ . Conversely, given an open map  $i$  that is also a regular monomorphism then it is equivalent to the open sublocale given by  $\exists_i(1_{\Omega X_0})$ . ■

Therefore, closed sublocales are parallel to open sublocales.

## 8 Pullback Stability of Proper and Open Maps

The following two theorems are parallel, showing the pullback stability of proper and open maps via identical techniques.

**Theorem 16** *If*

$$\begin{array}{ccc} W & \xrightarrow{p_2} & Y \\ p_1 \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

*is a pullback diagram in **Loc** and  $g$  is proper then*

- (i)  *$p_1$  is proper and*
- (ii)  *$\forall_{p_1} \Omega p_2(b) = \Omega f \forall_g(b) \quad \forall b \in \Omega Y$  (Beck-Chevalley)*

**Proof.**  $\Omega W$  is isomorphic to

$\mathbf{PFrm}(a \odot b \in \Omega X \otimes_{PFrm} \Omega Y \text{ (qua preframe)} \mid (\Omega f(c) \vee a) \odot b = a \odot (\Omega g(c) \vee b) \mid \forall a \in \Omega X, b \in \Omega Y, c \in \Omega Z)$

Define

$$\begin{aligned} \forall_{p_1} : \Omega W &\longrightarrow \Omega X \\ a \odot b &\longmapsto a \vee \Omega f \forall_g(b) \end{aligned}$$

This clearly satisfies the ‘‘qua preframe’’ conditions in the presentation of  $\Omega W$  since  $\forall_g$  is a preframe homomorphism. Given any  $a \in \Omega X, b \in \Omega Y, c \in \Omega Z$  we need to check  $(\Omega f(c) \vee a) \vee \Omega f \forall_g(b) = a \vee \Omega f \forall_g(\Omega g(c) \vee b)$ . But this follows from the coFrobenius condition which is satisfied by  $\Omega g \dashv \forall_g$ . So  $\forall_{p_1}$  is well defined. Is it right adjoint to  $\Omega p_1$ ?

Now  $\forall a \in \Omega X, b \in \Omega Y$

$$\begin{aligned} \forall_{p_1} \Omega p_1(a) &= \forall_{p_1}(a \odot 0) \\ &= a \vee \Omega f \forall_g(0) \\ &\geq a \end{aligned}$$

and

$$\begin{aligned}
\Omega p_1 \forall_{p_1}(a \odot b) &= (a \vee \Omega f \forall_g(b)) \odot 0 \\
&= (a \odot 0) \vee (\Omega f \forall_g(b) \odot 0) \\
&= (a \odot 0) \vee (0 \odot \Omega g \forall_g b) \\
&\leq (a \odot 0) \vee (0 \odot b) = a \odot b
\end{aligned}$$

Hence  $\Omega p_1 \dashv \forall_{p_1}$ . We check the coFrobenius condition for this adjunction. i.e. for every  $a, \bar{a} \in \Omega X$  and every  $b \in \Omega Y$  we want  $\forall_{p_1}((a \odot b) \vee \Omega p_1(\bar{a})) = \bar{a} \vee \forall_{p_1}(a \odot b)$ . Well,

$$\begin{aligned}
LHS &= \forall_{p_1}((a \vee \bar{a}) \odot b) \\
&= (a \vee \bar{a}) \vee \Omega f \forall_g(b) \\
&= \bar{a} \vee (a \vee \Omega f \forall_g(b)) \\
&= \bar{a} \vee \forall_{p_1}(a \odot b).
\end{aligned}$$

Finally given  $b \in \Omega Y$

$$\begin{aligned}
\forall_{p_1} \Omega p_2(b) &= \forall_{p_1}(0 \odot b) \\
&= \Omega f \forall_g(b)
\end{aligned}$$

and so condition (ii) in the statement of the theorem is satisfied. ■

**Theorem 17** *If*

$$\begin{array}{ccc}
W & \xrightarrow{p_2} & Y \\
p_1 \downarrow & & \downarrow g \\
X & \xrightarrow{f} & Z
\end{array}$$

is a pullback diagram in **Loc** and  $g$  is open then

- (i)  $p_1$  is open and
- (ii)  $\exists_{p_1} \Omega p_2(b) = \Omega f \exists_g(b) \quad \forall b \in \Omega Y$  (Beck-Chevalley)

**Proof.**  $\Omega W$  is isomorphic to **Sup** $\langle a \otimes b \in \Omega X \otimes_{Sup} \Omega Y$  (qua suplattice)  $| (\Omega f(c) \wedge a) \otimes b = a \otimes (\Omega g(c) \wedge b) \quad \forall a \in \Omega X, b \in \Omega Y, c \in \Omega Z \rangle$

Define

$$\begin{aligned}
\exists_{p_1} : \Omega W &\longrightarrow \Omega X \\
a \otimes b &\longmapsto a \wedge \Omega f \exists_g(b)
\end{aligned}$$

This clearly satisfies the ‘‘qua suplattice’’ conditions in the presentation of  $\Omega W$  since  $\exists_g$  is a suplattice homomorphism. Given any  $a \in \Omega X, b \in \Omega Y, c \in \Omega Z$  we need to check  $(\Omega f(c) \wedge a) \wedge \Omega f \exists_g(b) = a \wedge \Omega f \exists_g(\Omega g(c) \wedge b)$ . But this follows from the Frobenius condition which is satisfied by  $\exists_g \dashv \Omega g$ . So  $\exists_{p_1}$  is well defined. Is

it left adjoint to  $\Omega p_1$ ?  
 Now  $\forall a \in \Omega X, b \in \Omega Y$

$$\begin{aligned}\exists_{p_1} \Omega p_1(a) &= \exists_{p_1}(a \otimes 1) \\ &= a \wedge \Omega f \exists_g(1) \\ &\leq a\end{aligned}$$

and

$$\begin{aligned}\Omega p_1 \exists_{p_1}(a \otimes b) &= (a \wedge \Omega f \exists_g(b)) \otimes 1 \\ &= (a \otimes 1) \wedge (\Omega f \exists_g(b) \otimes 1) \\ &= (a \otimes 1) \wedge (1 \otimes \Omega g \exists_g b) \\ &\geq (a \otimes 1) \wedge (1 \otimes b) = a \otimes b\end{aligned}$$

Hence  $\exists_{p_1} \dashv \Omega p_1$ . We check the Frobenius condition for this adjunction. i.e. for every  $a, \bar{a} \in \Omega X$  and every  $b \in \Omega Y$  we need  $\exists_{p_1}((a \otimes b) \wedge \Omega p_1(\bar{a})) = \bar{a} \wedge \exists_{p_1}(a \otimes b)$ . Well

$$\begin{aligned}LHS &= \exists_{p_1}((a \wedge \bar{a}) \otimes b) \\ &= (a \wedge \bar{a}) \wedge \Omega f \exists_g(b) \\ &= \bar{a} \wedge (a \wedge \Omega f \exists_g(b)) \\ &= \bar{a} \wedge \exists_{p_1}(a \otimes b)\end{aligned}$$

Finally given  $b \in \Omega Y$

$$\begin{aligned}\exists_{p_1} \Omega p_2(b) &= \exists_{p_1}(1 \otimes b) \\ &= \Omega f \exists_g(b)\end{aligned}$$

and so condition (ii) in the statement of the theorem is satisfied. ■

## 9 Proper and Open Surjections

Recall that a locale map  $f : X \rightarrow Y$  is a surjection iff it is an epimorphism (iff  $\Omega f$  is an inclusion, i.e. 1-1, since the free frame on the singleton set can be defined). Clearly proper and open maps are closed under composition. As a partial converse we have that open and proper maps interact with surjections in the following useful way:

**Lemma 18** (i) *If  $X, Y, Z$  are locales and  $X \xrightarrow{q} Y \xrightarrow{f} Z$  is such that  $f'$  ( $= f \circ q$ ) is proper and  $q$  is a surjection then  $f$  is proper.*

(ii) *If  $X, Y, Z$  are locales and  $X \xrightarrow{q} Y \xrightarrow{f} Z$  is such that  $f'$  ( $= f \circ q$ ) is open and  $q$  is a surjection then  $f$  is open.*

**Proof.** (i) Define

$$\begin{aligned}\forall_f : \Omega Y &\rightarrow \Omega Z \\ y &\mapsto \forall_{f'} \Omega qy\end{aligned}$$

(ii) Define

$$\begin{aligned}\exists_f : \Omega Y &\rightarrow \Omega Z \\ y &\mapsto \exists_{f'} \Omega qy\end{aligned}$$

■

That any factorization of a proper map through a surjection gives another proper map in this way will be used when proving that image factorizations exist for compact Hausdorff locales. When dealing with proper or open maps, there is a simple characterization of surjectivity:

**Lemma 19** (i) *Given a proper map  $f : X \rightarrow Y$  then  $f$  is a surjection iff  $\forall_f(0) = 0$ ,*

(ii) *given an open map  $f : X \rightarrow Y$  then  $f$  is a surjection iff  $\exists_f(1) = 1$ .*

**Proof.** Immediate from the coFrobenius and Frobenius conditions respectively. ■

**Theorem 20** *Proper and open surjections are pullback stable.*

**Proof.** This is immediate from the description of proper and open surjections given in the lemma and the Beck-Chevalley equations derived alongside the proofs of the previous section which established the pullback stability of proper and open maps. ■

Proper and open surjections are *regular* epimorphisms and these facts again have identical proofs:

**Proposition 21** *If  $p : X \rightarrow Z$  is a proper surjection then*

$$X \times_Z X \begin{array}{c} \xrightarrow{p_1} \\ \rightarrow \\ \xrightarrow{p_2} \end{array} X \xrightarrow{p} Z$$

*is a coequalizer diagram in **Loc**.*

**Proof.**  $pp_1 = pp_2$  by the definition of pullback. Thus all we need to do is show that any  $f : X \rightarrow W$  with  $fp_1 = fp_2$  factors uniquely through  $p : X \rightarrow Z$ . Say  $\Omega p_1 \Omega f = \Omega p_2 \Omega f$ . It is sufficient to prove  $\forall_p : \Omega X \rightarrow \Omega Z$  has  $\Omega p \forall_p \Omega f c = \Omega f c$  for every  $c \in \Omega W$ . (Recall that  $\forall_p \Omega p(a) = a \ \forall a$  since  $p$  is a proper surjection.) Hence it is sufficient to check that  $\Omega p \forall_p u = u$  for any  $u$  with  $\Omega p_1 u = \Omega p_2 u$ . For any such  $u$  we have

$$\begin{aligned}\Omega p \forall_p u &= \forall_{p_1} \Omega p_2 u \text{ (Beck-Chevalley)} \\ &= \forall_{p_1} \Omega p_1 u = u\end{aligned}$$

The last line is because  $\Omega p_1$  is a proper surjection since it is the pullback of a proper surjection. ■

**Proposition 22** *If  $p : X \rightarrow Z$  is an open surjection then*

$$X \times_Z X \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} X \xrightarrow{p} Z$$

*is a coequalizer diagram in **Loc**.*

**Proof.**  $pp_1 = pp_2$  by definition of pullback, hence all we need to do is show that any  $f : X \rightarrow W$  with  $fp_1 = fp_2$  factors through  $p : X \rightarrow Z$ . So  $\Omega p_1 \Omega f = \Omega p_2 \Omega f$  and it is sufficient to prove  $\exists_p : \Omega X \rightarrow \Omega Z$  satisfies  $\Omega p \exists_p \Omega f c = \Omega f c$  for every  $c$ . Hence it is sufficient to show  $\Omega p \exists_p u = u$  for any  $u$  with  $\Omega p_1 u = \Omega p_2 u$ .

$$\begin{aligned} \Omega p \exists_p u &= \exists_{p_1} \Omega p_2 u \text{ (Beck–Chevalley)} \\ &= \exists_{p_1} \Omega p_1 u \\ &= u \end{aligned}$$

The last line is because  $\Omega p_1$  is a surjective open as it is the pullback of a surjective open. ■

## 10 Compact Hausdorff and Discrete Locales

Recall that a locale is said to be regular if any open,  $a$ , is the join of opens well inside  $a$ . An open  $b$  is well inside another open  $a$  (denoted  $b \triangleleft a$ ) iff there exists a third open  $c$  such that  $c \wedge b = 0$  and  $c \vee a = 1$ . See the beginning of Chapter III in [Johnstone 82] for a description of how this coincides with the usual definition of regularity for a topological space. Importantly, classically, the compact regular locales are exactly the compact Hausdorff spaces as there is an equivalence between the two categories assuming the prime ideal theorem.

Recall also that a locale is discrete iff its frame of opens is the power set of some set. The following two results were originally shown in [Vermeulen 91] and [JoyTie 84] respectively:

**Proposition 23** (i) *A locale  $X$  is compact regular iff  $! : X \rightarrow 1$  and  $\Delta : X \rightarrow X \times X$  are proper.*

(ii) *A locale  $X$  is discrete iff  $! : X \rightarrow 1$  and  $\Delta : X \rightarrow X \times X$  are open.*

**Proof.** (i) There is a proof based on the preframe techniques used here in [Townsend 96], Theorem 3.4.2.

(ii) Chapter V, Section 5 of [JoyTie 84]. ■

Because of the pullback stability results, it is clear that a locale is discrete iff its finite diagonals are open and is compact regular (classically compact Hausdorff) iff its finite diagonals are proper. Given this classical correspondence between compact regular and compact Hausdorff we shall often use the expression “compact Hausdorff locale” to mean exactly the same thing as “compact regular locale”. We therefore have:

*Compact Hausdorff is parallel to discrete.*

It is well known that the category of compact Hausdorff spaces is regular, as is (rather trivially) the category of discrete spaces. Both these facts can re-emerge from a single proof using either proper or open maps:

**Proposition 24** (i) *The full subcategory, **KRegLoc**, of compact Hausdorff locales is regular.*

(ii) *The full subcategory, **DisLoc**, of discrete locales is regular.*

**Proof.** (i) (Outline proof only; details are in the final section of Chapter 3 in [Townsend 96].) A category is regular iff it has finite limits and pullback stable image factorizations. Closure under binary product can be seen since (i)  $\Delta^{X \times Y}$  is the pullback of  $\Delta^X \times \Delta^Y$  where  $\Delta^W$  denotes the diagonal of the locale  $W$  (by pullback stability and stability under composition if  $f, g$  are proper then so is  $f \times g$ ) and (ii)  $! : X \times Y \rightarrow 1$  is the composition  $X \times Y \xrightarrow{\pi_1} X \rightarrow 1$  and  $\pi_1$  is proper as it is the pullback of a proper map. Similarly for equalizers. For image factorization recall that any locale map can be factored as a surjection followed by a sublocale. Pulling back the diagonal along the sublocale gives a proper diagonal and applying Lemma 18 with the surjection shows that the image factorization is within the category of compact regular locales. (In detail, if  $f : X \rightarrow Y$  factors as  $X \xrightarrow{q} f[X] \hookrightarrow Y$  then apply Lemma 18 with  $f' \equiv !^X : X \rightarrow 1$  and  $f \equiv !^{f[X]} : f[X] \rightarrow 1$  to prove that  $f[X]$  is compact.)

Any locale map  $f : Y \rightarrow X$  can be factored as  $Y \xrightarrow{(1,f)} Y \times X \xrightarrow{\pi_2} X$  and these components will be proper if  $X, Y$  are compact regular. Therefore the surjective part of the image factorization just described is a proper map and so is a pullback stable regular epimorphism.

(ii) Identical proof structure given the characterization of discrete locales in terms of open finite diagonals. ■

## 11 Allegories

The parallel categories (compact regular/discrete) are regular and therefore for each, via the well known results of Freyd and Šcedrov ([FreydSced 90]) one can form an allegory (a type of category) whose objects are the same and whose morphisms are relations. Composition is given by relational composition. Key to our development a formula exists that expresses this relational composition in terms of an operation on the corresponding frame of opens. On the suplattice side:

**Lemma 25** *There is a bijection between the open sublocales of  $X \times Y$  for discrete  $X, Y$  and suplattice homomorphisms from  $\Omega X$  to  $\Omega Y$ . Relational composition is sent to function composition under this bijection.*

This can be proved easily by using the definition of discrete and suplattice. The preframe parallel is a little harder to prove (see Chapter 5 of [Townsend 96] for details):

**Lemma 26** *There is a bijection between the closed sublocales of  $X \times Y$  for compact Hausdorff  $X, Y$  and preframe homomorphisms from  $\Omega X$  to  $\Omega Y$ . Relational composition is sent to function composition under this bijection.*

These lemmas can also be viewed as results about the lower and upper power locale constructions respectively. We have a parallel set of definitions:

**Definition 27** (i) *Given a locale  $X$  the lower power locale on  $X$ , denoted by  $P_L(X)$ , is defined by*

$$\Omega P_L X \equiv \mathbf{Frm}\langle \Omega X \text{ qua suplattice} \rangle,$$

(ii) *given a locale  $X$  the upper power locale on  $X$ , denoted by  $P_U(X)$ , is defined by*

$$\Omega P_U X \equiv \mathbf{Frm}\langle \Omega X \text{ qua preframe} \rangle.$$

Clearly these define two monads on the category of locales via the adjunction given by constructing free frames qua, respectively, suplattices and preframes. In essence therefore the preceding two lemmas are saying:

**Proposition 28** (i)  *$REL(\mathbf{DisLoc}) \cong \mathbf{DisLoc}_{P_L}$  and*

(ii)  *$REL(\mathbf{KRegLoc}) \cong \mathbf{KRegLoc}_{P_U}$ ,*

*where  $REL(-)$  is the construction of an allegory of objects and relations on a regular category and the right hand sides are the Kleisli categories.*

## 12 Hausdorff Systems

In [Vickers 93] Vickers introduces the category of continuous information systems (**Infosys**) as models applicable to the denotational semantics of computer programs. Continuous information systems are defined as pairs  $(X, R)$  where  $X$  is a set and  $R$  is a relation on  $X$  which is idempotent with respect to relational composition. There are many morphisms possible between continuous information systems. The most general are relations:

$$R : (X, R_X) \rightarrow (Y, R_Y)$$

where  $R \subseteq X \times Y$  is such that  $R = R_Y \circ R \circ R_X$  and  $\circ$  is relational composition. These are called the lower semicontinuous approximable mappings. An equivalent description is to say that **Infosys** is the Karoubi envelope on  $REL(\mathbf{DisLoc}) \cong \mathbf{DisLoc}_{P_L}$ .

So it is natural to introduce the parallel of continuous information systems as follows: the category of *Hausdorff systems* (denoted **HausSys**) is the Karoubi envelope on  $REL(\mathbf{KRegLoc}) \cong \mathbf{KRegLoc}_{P_U}$ . In other words a Hausdorff system is a pair  $(X, R)$  where  $X$  is a compact Hausdorff locale and  $R$  is a closed relation such that  $R \circ R = R$ .

If  $(X, R)$  is an infosys, then we know (Lemma 25) that there is a suplattice homomorphism  $\downarrow^R: PX \rightarrow PX$  corresponding to  $R$ .  $\downarrow^R$  is idempotent since  $R$  is. The set

$$\{T \mid T \in PX \quad \downarrow^R T = T\}$$

can then be seen to be a completely distributive lattice. The essence of [Vickers 93] is a proof that all completely distributive lattices arise in this way. Given a Hausdorff system  $(X, R)$  we know that there is a preframe homomorphism  $\Downarrow^{op}: \Omega X \rightarrow \Omega X$  corresponding to  $R$  which is idempotent (Lemma 26). The notation ‘*op*’ reflects the fact that closed sublocales are in order isomorphism with  $\Omega X^{op}$ ; as an action on closed sublocales, i.e. spatially, we are taking a lower closure,  $\Downarrow$ , with respect to  $R$ . As an action on opens we therefore adopt  $\Downarrow^{op}$ . Notice that,

$$\{a \mid a \in \Omega X \quad \Downarrow^{op} a = a\}$$

is a subpreframe of  $\Omega X$ . It has finite joins:  $\Downarrow^{op} 0$  is least and the join of  $a, b$  is given by  $\Downarrow^{op} (a \vee b)$ . Further,

**Lemma 29**  $\Omega \bar{X} \equiv \{a \mid a \in \Omega X \quad \Downarrow^{op} a = a\}$  is the frame of opens of a stably locally compact locale.

Although it is well known what the completely distributive lattices are (for instance we can view them spatially as just the continuous posets), it is perhaps less well known what the stably locally compact locales are. Banaschewski and Brümmer describe them as corresponding to the “most reasonable not necessarily Hausdorff compact spaces”. Johnstone ([Johnstone 82]) captures them as exactly the retracts in **Loc** of the coherent locales. The standard definition is a locale whose frame of opens is a stably continuous lattice, i.e. the directed join map from the ideal completion of the frame to itself has a left adjoint, and that left adjoint preserves finite meets. Or, to put this another way, (i) every open is the join of opens way below ( $\ll$ ) it, (ii)  $1 \ll 1$  and (iii)  $a \ll b_1, b_2$  implies  $a \ll b_1 \wedge b_2$ . (Recall that  $a \ll b$  iff whenever  $b \leq \bigvee^\uparrow T$  then there exists  $t \in T$  such that  $a \leq t$ , and that an ideal is a lower closed directed subset; the ideal completion is the set of all ideals.)

**Proof.** First we check that the frame is continuous i.e. that

$$\forall a \in \Omega \bar{X} \quad a = \bigvee^\uparrow \{b \mid b \ll_{\Omega \bar{X}} a\} \quad (*).$$

Since  $\Omega X$  is compact regular we know that  $(\forall a, b \in \Omega X)$

$$a \triangleleft b \quad \Leftrightarrow \quad a \ll b$$

hence to conclude (\*) all we need do is check that

$$b \ll a \quad \Rightarrow \quad \Downarrow^{op} b \ll_{\Omega \bar{X}} a$$

if  $a \in \Omega\bar{X}$ . Say  $b \ll a$  and  $a \leq \bigvee^\uparrow S$   $S \subseteq^\uparrow \Omega\bar{X}$  then  $\exists s \in S$   $b \leq s \Rightarrow \Downarrow^{op} b \leq \Downarrow^{op} s = s$ .

As for stability we need to check that  $1 \ll_{\Omega\bar{X}} 1$  (trivial by compactness of  $\Omega X$ ) and  $a \ll_{\Omega\bar{X}} b_1, b_2$  implies  $a \ll_{\Omega\bar{X}} b_1 \wedge b_2$ . Since  $b_i \in \Omega\bar{X}$ ,  $\Omega X$  is regular and  $\Downarrow^{op}$  is a preframe homomorphism we know that

$$b_i = \bigvee^\uparrow \{\Downarrow^{op} c \mid c \triangleleft b_i\}$$

Hence  $a \leq \Downarrow^{op} c_i$  for some  $c_1, c_2$  with  $c_i \triangleleft b_i$ . Hence  $a \leq \Downarrow^{op} (c_1 \wedge c_2)$ . But  $c_1 \wedge c_2 \triangleleft b_1 \wedge b_2$  and so  $c_1 \wedge c_2 \ll b_1 \wedge b_2$ . Hence  $a \ll_{\Omega\bar{X}} b_1 \wedge b_2$ .  $\blacksquare$

The aim for the rest of this section on Hausdorff systems is to prove that every stably locally compact locale arises in this way. The proof is a manipulation of Banaschewski and Brümmer's proof that stably locally compact locales are dual to compact regular biframes [BanaBrüm 88], though we do not use any bitopological techniques. A more detailed account can be found in the final Chapter of [Townsend 96].

Let  $\mathbf{StLocKLoc}_{P_U}$  be the category whose objects are stably locally compact locales and morphisms are all (formally reversed) preframe homomorphisms. Bearing in mind the correspondence between preframe homomorphisms on the frame of opens of compact Hausdorff locales and closed relations on these locales it should be clear that there is a functor:

$$\begin{aligned} \mathcal{C} : \mathbf{HausSys} &\rightarrow \mathbf{StLocKLoc} \\ (X, R) &\mapsto \bar{X} \end{aligned}$$

where  $\Omega\bar{X} = \{a \in \Omega X \mid \Downarrow^{op} a = a\}$ . We wish to define

$$\mathcal{B} : \mathbf{StLocKLoc} \rightarrow \mathbf{HausSys}$$

such that  $\mathcal{CB}(X) \cong X$ . Say  $X$  is a given stably locally compact locale. Define  $\Lambda\Omega X$  to be the set of Scott open filters of  $\Omega X$ . So for any  $F \subseteq \Omega X$ ,  $F \in \Lambda\Omega X$  iff

- (i)  $F$  is upper closed,
- (ii)  $a, b \in F \Rightarrow a \wedge b \in F$ ,
- (iii)  $1 \in F$  and
- (iv)  $a \in F \Rightarrow \exists b \in F$   $b \ll a$ .

For example  $\uparrow a \in \Lambda\Omega X$  for every  $a \in \Omega X$  as  $\ll$  is interpolative since  $a = \bigvee^\uparrow \{b \mid \exists \bar{a}, b \ll \bar{a} \ll a\}$ .

**Lemma 30**  $\Lambda\Omega X$  is a stably locally compact locale.

**Proof.** Directed join is given by union and finite meet is intersection. Finite join is given by

$$F_1 \vee F_2 = \uparrow \{a_1 \wedge a_2 \mid a_1 \in F_1, a_2 \in F_2\}.$$

Note that for all  $F \in \Lambda\Omega X$

$$F = \bigcup^\uparrow \{\uparrow a \mid a \in F\}$$

and  $\uparrow a \ll_{\Lambda\Omega X} F$  for all  $a \in F$ . ■

Now since  $X$  is stably locally compact we know that there is a frame injection  $\downarrow : \Omega X \rightarrow \text{Idl}\Omega X$ . Define  $B_{\Omega X}$  to be the free Boolean algebra on  $\Omega X$  qua distributive lattice (this can be done via finitary universal algebra, though there is a proof in Ch. 1 (1.3) of [Townsend 96] that does not use a natural numbers object). There is a frame injection of  $\text{Idl}\Omega X$  into  $\text{Idl}B_{\Omega X}$  so by composing this injection with  $\downarrow$  we find that  $\Omega X$  can be embedded in  $\text{Idl}B_{\Omega X}$ .

**Lemma 31**  $\Lambda\Omega X$  can be embedded into  $\text{Idl}B_{\Omega X}$ .

**Proof.** Send  $F$  to  $\bigcup_{b \in F}^\uparrow \downarrow \neg b$ . It is routine to check that this is a frame injection. ■

**Define:**  $\Omega Y =$  the subframe of  $\text{Idl}B_{\Omega X}$  generated by the image of the above two embeddings.

**Theorem 32**  $Y$  is a compact Hausdorff locale.

**Proof.** Compactness is immediate since  $\Omega Y$  is a subframe of the compact frame  $\text{Idl}B_{\Omega X}$ . As for regularity it is sufficient to check that

$$\downarrow a = \bigcup^\uparrow \{I \mid I \triangleleft \downarrow a\}$$

for every  $a \in \Omega X$  and

$$\bigcup_{b \in F}^\uparrow \downarrow \neg b = \bigvee^\uparrow \{I \mid I \triangleleft \bigcup_{b \in F}^\uparrow \downarrow \neg b\}$$

$\forall F \in \Lambda\Omega X$ .

However  $a = \bigvee^\uparrow \{x \mid x \ll a\}$  and  $F = \bigvee^\uparrow \{G \mid G \ll F\}$  since both  $\Omega X$  and  $\Lambda\Omega X$  are continuous posets. So it is sufficient to prove that

$$x \ll a \Rightarrow \downarrow x \triangleleft \downarrow a \quad (\text{I})$$

$$G \ll F \Rightarrow \bigcup_{b \in G}^\uparrow \downarrow \neg b \triangleleft \bigcup_{b \in F}^\uparrow \downarrow \neg b \quad (\text{II})$$

**(I):** Say  $x \ll a$ . Set  $F = \uparrow x$  (a Scott open filter). Then  $\bigcup_{b \in F}^\uparrow \downarrow \neg b \in \Omega Y$ . But clearly

$$\downarrow x \cap \bigcup_{b \in F}^\uparrow \downarrow \neg b = 0.$$

Further  $x \ll a \Rightarrow \exists \bar{a} \ x \ll \bar{a} \ll a$ . Hence

$$\begin{aligned} \downarrow a \vee \bigcup_{b \in F}^{\uparrow} \downarrow \neg b &\geq \downarrow a \vee \downarrow \neg \bar{a} \\ &\geq \downarrow \bar{a} \vee \downarrow \neg \bar{a} = 1. \end{aligned}$$

Hence  $\downarrow x \triangleleft \downarrow a$ .

(II): Say  $G \ll F$ . So  $\exists x \in F \ G \subseteq \uparrow x \subseteq F$  (since  $F = \bigcup^{\uparrow} \{\uparrow x \mid x \in F\}$ ). Then

$$\bigcup_{b \in G}^{\uparrow} \downarrow \neg b \cap \downarrow x = 0.$$

Now  $x \in F \Rightarrow \exists \bar{x} \in F \ \bar{x} \ll x$  and so

$$\downarrow x \vee \bigcup_{b \in F}^{\uparrow} \downarrow \neg b \geq \downarrow \bar{x} \vee \downarrow \neg \bar{x} = 1.$$

■

We want a closed idempotent relation on  $Y$  and so need to find a preframe endomorphism  $\Downarrow^{op}: \Omega Y \rightarrow \Omega Y$  such that  $\Downarrow^{op2} = \Downarrow^{op}$ . If  $I, J \in \Omega Y$  we write  $I \bar{\triangleleft} J$  if and only if  $\exists F \in \Lambda \Omega X$  such that

$$\begin{aligned} I \cap \bigcup_{b \in F}^{\uparrow} \downarrow \neg b &= 0 \\ J \vee \bigcup_{b \in F}^{\uparrow} \downarrow \neg b &= 1 \end{aligned}$$

Clearly  $\bar{\triangleleft} \subseteq \triangleleft$  and the last proof has shown us that  $x \ll a$  implies  $\downarrow x \bar{\triangleleft} \downarrow a$ . Define

$$\begin{aligned} \Downarrow^{op}: \Omega Y &\rightarrow \Omega Y \\ J &\mapsto \bigcup^{\uparrow} \{\downarrow a \mid \text{some } a \bar{\triangleleft} J\}. \end{aligned}$$

Facts about  $\Downarrow^{op}$ :

- \*  $\forall J, \Downarrow^{op}(J) = \downarrow a$  for some  $a \in \Omega X$
- \*  $\Downarrow^{op}(\downarrow a) = \downarrow a \ \forall a$
- \*  $(\Downarrow^{op})^2 = \Downarrow^{op}$
- \*  $\Downarrow^{op}$  is a preframe homomorphism.

Hence define  $\mathcal{B}: \mathbf{StLocKLoc} \rightarrow \mathbf{HausSys}$  by  $\mathcal{B}(X) = (Y, R)$ , where  $R$  is the closed relation corresponding to  $\Downarrow^{op}$ . It should be clear from construction that  $\mathcal{CB}(X) \cong X$ .

**Remark 33** (a) *This proof is not parallel in structure to Vickers' proof that the continuous information systems are exactly the completely distributive lattices. This is true also of the proofs of Proposition 23. The parallel does not extend to describing separation axioms in their more well known forms. We rely on ad-hoc proofs (such as the above) to manipulate well known separation axioms into statements about proper/open maps. Once translated to statements about*

proper/open maps then the parallel is transparent. The information/Hausdorff systems are both Karoubi envelopes.

(b) There is an alternative description of the  $\mathcal{B}$  functor given in Escardó's paper [Escardó 02]. The  $\mathcal{B}$  functor is better known as the Patch construction; our equivalence does extend Priestley duality, see Chapter 8 of [Townsend 96].

(c) Hoffman-Lawson duality for Hausdorff systems is immediate. For any Hausdorff system its Hoffman-Lawson dual is found by twisting the closed idempotent relation. An account of Hoffman-Lawson duality is contained in VII, 2 of [Johnstone 82].

## 13 Conclusions

We have not formalized the parallel between the preframe and suplattice approaches to locale theory but have argued with the following examples:

$$f : X \rightarrow Y \text{ proper} \quad f : X \rightarrow Y \text{ open}$$

$X$ compact binary Tychonoff proper (surjections) are pullback stable proper surjections are reg. epi. $X$ compact regular <b>KRegLoc</b> a regular category upper power locale, $P_U$ Hausdorff systems	$X$ open $X, Y$ open implies $X \times Y$ open open (surjections) are pullback stable open surjections are reg. epi. $X$ discrete <b>DisLoc</b> a regular category lower power locale, $P_L$ information systems
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They are parallel by replacing finite joins with finite meets and preframes with suplattices. The parallel would not be as clearly observable were we to work in a classical context since the excluded middle would imply that all locales are open.

## 14 Notes on Results

It was with [JohVic 91] that the basic lattice theoretic properties of preframes were developed (though see [Banaschewski 88]). Then, in [Vermeulen 93], it became clear that preframes could be used when working with proper maps in general. The results of the paper [Vermeulen 93] very much mimic the results of [JoyTie 84], and taking the similarities in the lattice theoretic structures (suplattice/preframe) very much to heart, in [Townsend 96], the results of [Vermeulen 93] and [JoyTie 84] are reproved side by side showing the similarities in the proof structure. This paper is essentially an “edited highlights”

of [Townsend 96]. Also in [Vickers 95] important steps are made towards a categorical abstraction for a unifying theory of which the theories of proper and open maps are examples.

That frame coproduct is given by preframe tensor (Theorem 1, (i)) is in [JohVic 91] and the suplattice version, (ii), is in [JoyTie 84]. The key definitions of open and proper maps (covered by Definition 12) are in [JoyTie 84] and [Vermeulen 93] respectively. The description of frame coequalizer in terms of suplattices (Thm 3, (i)) is implicit in Johnstone’s construction of frames via C-ideals in his coverage theorem ([Johnstone 82]) and explicit in [AbrVic 93]. The preframe version (Thm 3, (ii)) is immediate from the coverage theorem in [JohVic 91]. The creation of filtered colimits (Prop. 6) is a well established categorical result and is explicit in [JohVic 91]. The proper pullback stability result (Thm 16) is originally in [Vermeulen 93] and the open pullback stability result is in [JoyTie 84]. Similarly for the results of Prop. 21 and 22 respectively (proper/open surjections are regular epis). Regularity of the categories **KRegLoc** and **DisLoc** (Prop. 24) via parallel techniques is new to [Townsend 96], though that they are regular is of course well known. The description of the corresponding allegories in terms of Kleisli categories was initially shown in [Vickers 97]. The final section on Hausdorff systems reports on work that is new in [Townsend 96].

## 15 Further Work

Since this work on the relationship between preframe and suplattice approaches was written up in e.g. [Vermeulen 93], [Vickers 95] and [Townsend 96], further results have been developed which step towards answering the obvious background question: Can one formalize the relationship? The parallel, as stated and developed here, is only argued by example. Intuitively, the parallel would become a formal duality provided we could somehow take the dual of the finitary lattice structure without disturbing the infinitary (directed join) part. This may be possible given the following technical lemma observed in [TowVic 02].

**Lemma 34** *There is a bijection between natural transformations  $\mathbf{Loc}(\_ \times X, \mathbb{S}) \rightarrow \mathbf{Loc}(\_ \times Y, \mathbb{S})$  in  $[\mathbf{Loc}^{op}, \mathbf{Set}]$  and directed join preserving maps from  $\Omega X$  to  $\Omega Y$ .*

( $\mathbb{S}$ , the Sierpiński locale, is defined by  $\Omega \mathbb{S} = \mathbf{Frm}\langle 1 \rangle$ , i.e. the free frame on the singleton set.  $\mathbb{S}$  is an internal distributive lattice in **Loc**.) This provides an external categorical description of the part of the theory of spaces that we would like to remain fixed under any proposed proper/open duality. Further, this lemma specializes: suplattice homomorphisms are exactly those natural transformations that preserve the join semilattice structure implied by  $\mathbb{S}$  and preframe homomorphisms are exactly those natural transformations that preserve the meet semilattice structure implied by  $\mathbb{S}$ . The duality is therefore “treat the relevant maps as natural transformations and dualize the order on  $\mathbb{S}$ ”.

To discover exactly what fragment of the theory of locales is dual under the proper/open duality it is probably easiest to axiomatize an abstract category of

spaces for which the duality is immediate. **Loc** will then be an example of an abstract category of spaces. Any truth implied by the axioms of the abstract category of spaces will automatically have a proper/open dual. In this way it is that fragment of locale theory which is derivable from only these axioms that will always have a proper/open dual. The suggested axioms for such an abstract category of spaces **C** is

- (i) **C** is order enriched,
- (ii) there exists an internal distributive lattice in **C**, denoted **S**, classifying (via top and bottom points) closed and open regular subobjects in **C**,
- (iii) for every abstract space  $X$ , there exists another space  $\mathbb{P}X$  such that  $\mathbf{C}(Y, \mathbb{P}X) \cong \text{Nat}[\mathbf{C}(\_ \times X, \mathbb{S}), \mathbf{C}(\_ \times Y, \mathbb{S})]$  naturally in  $Y$  and
- (iii') (iii) is stably true (i.e. true under slicing).

The space  $\mathbb{P}X$  is taking the role of the double power locale, i.e. the composition, in either order, of the lower power locale followed by the upper power locale. The double power locale construction, by definition, describes directed join preserving maps between frames in terms of locale maps. See [JohVic 91] for background or [Vickers 02] for substantial work on the double power locale. Given these axioms the proper/open duality is found by reversing the order enrichment.

These axioms are not yet complete and will be the subject of further work. For instance clarity is needed on what limits/colimits exist in **C** and how they distribute. Initial analysis is available in [Townsend 04].

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