Localic Priestley Duality

Christopher F. Townsend

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Abstract

Given the category of ordered Stone spaces (as introduced by Priestley) and the cateogry of coherent spaces (= spectral spaces) we can construct a pair of functors

$$\mathsf{CohSp} \underset{\mathcal{C}}{\overset{\mathcal{B}}{\rightleftarrows}} \mathsf{OStoneSp}$$

between the categories. Priestley [Pri70] has shown, assuming the prime ideal theorem, that these define an equivalence. In this paper we define ordered Stone locales. These are classically just the ordered Stone spaces. It is well known that the localic analogue for the coherent spaces is the category of coherent locales. We prove, entirely constructively, that the category of coherent locales is equivalent to the category of ordered Stone locales.

1 Introduction

The objective of this paper is to give an entirely constructive version of a proof of Priestley's duality:

$$\mathsf{OStoneSp} \cong \mathsf{CohSp}$$

where OStoneSp is the category of ordered Stone space and CohSp are the coherent (or spectral) spaces. It seems odd to use the word 'duality' but we are simply assuming that the reader is familiar with the duality

$$\mathsf{CohSp} \cong \mathsf{Dlat}^{op}$$

and so we are viewing Priestley's duality as an extension of this well known (Stone) duality.
It is easy to construct functors between the category OStoneSp and CohSp. Proving them to be equivalent strictly requires the prime ideal theorem. We shall not repeat Priestley's proof here, but we will outline how an assumption of its conclusion implies PIT.

It would therefore appear futile to try to prove a constructive version of this theorem. Hence we work with locales instead of spaces and find that the localic analogue of the result is constructively valid. i.e. valid in any topos.

Classically we will see that the category of ordered Stone locales is equivalent to the category of ordered Stone spaces. The localic analogue of a coherent space is just a coherent locale and the classical fact that the localic and spatial analogues are equivalent is just another way of looking at Stone duality.

The main part of this work is an entirely constructive proof that the category of ordered Stone locales is equivalent to the category of coherent locales. By 'entirely constructive' we simply mean that no use is made of the excluded middle or PIT. We mean Kuratowski finite when we use the word finite. Although an intuitive idea of what finite means should see the reader through it is worth noting that, informally, a Kuratowski finite set is any set for which there exists a finite listing of the elements. Since in this constructive context inequality is not necessarily decidable we allow repititions in our list.

The proof that ordered Stone locales are equivalent to coherent locales is not entirely straight forward. For a start it is not immediately obvious what a localic poset should be since the transitive rule

$$x \leq y, \quad y \leq z \quad \Rightarrow \quad x \leq z$$

cannot be easily expressed as a fact about locales since it refers to points. Another way of stating this transitive rule is

$$(\leq);(\leq) \subseteq (\leq)$$

where ; is relational composition. There is no clear way of defining relational composition of arbitary localic relations (=sublocales of binary products of locales), however we find that we can define relational composition on closed relations of compact regular locales. Once the preframe definition of the frame of opens of a product locale is understood we are able to define a formula for such a relational composition, and it is this formula that allows us to define what a localic poset is (provided the locale is compact regular and the relation is closed) and allows us to prove the equivalence of ordered Stone locales and coherent locales.

2 Priestley Duality

This section is an attempt to explain the background problem rather than a piece of self contained mathematical expostion. Consult Section II 4.5 - 4.9 of Stone Spaces [Joh82] for the more detailed account upon which this exposition is based.

An ordered Stone space is a compact topological poset which is totally order seperated. i.e. if $x \not\leq y$ then $\exists$ a clopen set U such that $\uparrow U = U$ and $x \in U$ and $y \notin U$. So if $x \neq y$ then they are seperated by some clopen. This means that the space is compact and totally seperated. Hence it is Stone. If it is Stone then it is compact Hausdorff and so a subset is closed iff it is compact. Hence a subset is clopen if and only if it is compact open. Notice that totally order seperated implies that $\leq$ is a closed subset of $X \times X$. And so we see that

Lemma 2.1 If $(X, \leq)$ is a compact topological poset then it is an ordered Stone space if and only if X is Stone, $\leq$ is closed and
$\not\leq = \bigcup\{U \otimes U^c | \uparrow U = U, U$ compact open.$\}$ □

We now turn to Priestley duality and define

$$\begin{aligned} \mathcal{B} : \mathsf{CohSp} &\longrightarrow \mathsf{OStoneSp} \\ (X, \Omega) &\longmapsto (X, \text{`patch'}, \leq) \end{aligned}$$

where the 'patch' topology is based by

$$\{U \cap V^c | U, V \text{ compact opens of } \Omega\}$$

and $\leq$ is the specialization order on (X, Ω). It can be shown that

Lemma 2.2 The set of compact opens of 'patch' forms the free Boolean algebra on the distributive lattice of compact opens of X. □

In the other direction we have

$$\begin{array}{rcl} \mathcal{C}:\mathsf{OStoneSp} & \longrightarrow & \mathsf{CohSp} \\ (X,\Omega,\leq) & \longmapsto & (X,\{U|U\in\Omega,\uparrow U=U\}) \end{array}$$

Lemma 2.3 $\{U|U\in\Omega,\uparrow U=U\}=Idl\{U|U\in\Omega,\uparrow U=U\}$. ($IdlA$ is the set of directed subsets of A.) i.e. $\mathcal{C}(X,\Omega\leq)$ is coherent.

In section II 4.9 of Stone Spaces [Joh82] Johnstone shows how an assumption that $\mathcal{BC}$ defines an equivalence allows us to conclude the PIT.

Let us assume that $\mathcal{B},\mathcal{C}$ define an equivalence. We see straight away that if a coherent space is T_1 (i.e. if the specialization order $\leq$ is equality) then it is Stone. But T_1ness can equivalently be defined as saying that all points are closed. For any distributive lattice A the points of the associated coherent space are the prime ideals and the closed points are the maximal ideals. Hence the statment of T_1ness is equivalent to the statement that the maximal and prime ideals coincide. But assuming $\mathcal{B},\mathcal{C}$ define an equivalence we know that a coherent space is T_1 if and only if it is Stone. Hence:

Lemma 2.4 (Nac49) A distributive lattice is Boolean if and only if all its prime ideals are maximal. □

To see that this lemma implies PIT is not immediately obvious. It certainly proves that any non-Boolean distributive lattice has a prime ideal. But any non-trivial Boolean can be embedded into a non-trivial non-Boolean distributive lattice and so we have PIT. To see how to construct such an embedding consult Exercise I 4.8 of Stone Spaces ([Joh82]).

3 Ordered Stone Locales

If X is a locale we write ΩX for the corresponding frame of opens. If $f:X\to Y$ is a locale map then $\Omega f:\Omega Y\to\Omega X$ is the corresponding frame homomorphism. If $Y\rightarrowtail X$ is a closed sublocale of X then $Y=\neg a$ for some $a\in\Omega X$. a is referred to as the open corresponding to the closed sublocale Y.

We want to look at localic posets. i.e. pairs $(X,\leq)$ where X is a locale and $\leq$ is some sublocale of $X\times X$. In view of the definition of ordered Stone space we will be restricting to the case where $\leq$ is closed. We also want $\leq$ to be a partial order. Clearly reflexivety is the statement that the diagonal (Δ) is less than $\leq$ in the poset of sublocales of $X\times X$ (=$Sub(X\times X)$). It is well known that $Sub(X\times X)$ has finite meets and so the anti-symmetry axiom for $\leq$ is just

$$(\leq)\wedge(\geq)\leq_{\mathsf{Sub}(X\times X)}\Delta$$

where $\geq$ is the composition of $\leq$ with the twist isomorphism of $X\times X$. Finally we have the problem of transitivety. As pointed out in the introduction we can write the transitivety axiom as

$$(\leq);(\leq)\leq(\leq)$$

where ; is relational composition. Also note that we only need to define relational composition on closed relations of compact regular locales. This is because Stone locales are compact regular and we are only examining closed relations. We leave aside till section 4 the definition of such a relational composition except to note that since the compact regular locales form a regular category we know that such a relational composition can be defined.

We want to define OStoneLoc, the category of ordered Stone locales. Given lemma 2.1 we clearly need to find a localic analog to the condition "a is upper closed" where a is some open of a Stone locale X. But an open set is upper closed iff its complement is lower closed. So we use the condition "$\neg a$ is lower closed" to replace the spatial intuition "a upper closed". Once we have relational composition we can define what "$\neg a$ lower closed" means: it is simply the statement

$$(\leq);(\neg a) \leq (\neg a) \qquad (*)$$

where ; is relational composition of closed sublocales. (The reader who is worried about the fact that $\neg a \rightarrowtail X$ is not a relation should note that $X \cong X \times 1$ and so $\neg a$ can be viewed as a relation on $X \times 1$.) The spatial intuition behind lower closure should then suffice to convince us that (*) does define what it means for a sublocale to be lower closed.

Definition:(cf lemma (2.1)) $(X, \leq)$ is an ordered Stone locale iff X is a Stone locale, i.e. $\Omega X = IdlK\Omega X$ where $K\Omega X$ (= the set of compact opens of ΩX) is Boolean, and $\leq$ is a closed partial order on X such that

$$a_{\leq} = \bigvee\{a \otimes \neg a | a \in K\Omega X, \Downarrow \neg a = a\}$$

Where $\Downarrow \neg a \equiv \leq; \neg a$.

We also want to define

$$\mathcal{C} : \mathsf{OStoneLoc} \longrightarrow \mathsf{CohSp}$$

and it should be clear from lemma 2.3 that the choice for $\mathcal{C}$ will be

$$\Omega\mathcal{C}X = Idl\{a | a \in K\Omega X, \Downarrow \neg a = \neg a\}$$

Clearly we would like a formula for relational composition.

4 Relational Composition

A preframe is a poset with finite meets and directed joins such that the directed joins distribute over the finite meets. It is known ([JV91]) that the category of preframes has a tensor and that the preframe tensor of two frames gives their coproduct. So if X, Y are two locales then

$$\Omega(X \times Y) = \Omega X \otimes \Omega Y$$

where $\otimes$ is preframe tensor. A typical generator of this tensor is written $a\mathrm{O}b \quad (a \in \Omega X, b \in \Omega Y)$.
Some spatial intuition behind this result can be found in the following: if X, Y are topological spaces and if for U open in X and V open in Y we define

$$U\mathrm{O}V \equiv \{(x, y) | x \in U \text{ or } y \in V\}.$$

Then the least subpreframe of $P(X \times Y)$ which contains all these sets is the product topology on $X \times Y$.
We continue with our spatial intuition. Say X, Y, Z are spaces and$R_1 \subseteq X \times Y, R_2 \subseteq Y \times Z$ are both closed. So $R_i = \neg I_i$ where $\neg$ is set complement and the I_i's are open.
We want $R_1; R_2$ to be closed and so to define ; all we need define is some function

$$* : \Omega(X \times Y) \times \Omega(Y \times Z) \to \Omega(X \times Z)$$

such that $R_1; R_2 = \neg * (I_1, I_2)$. Given the above facts about preframe tensors it should be clear that we only need be concerned with the cases

$$I_1 = U_1\mathrm{O}V_1 \qquad I_2 = V_2\mathrm{O}W_2$$

We know $(x,z) \in R_1; R_2$ iff $\exists y \quad xR_1y \quad yR_2z$. Hence $(x,z) \in *(I_1, I_2)$ iff $\forall y \quad \neg(xR_1y) \vee \neg(yR_2z)$. (We are only looking at the spatial case in order to justify the choice of formula to follow and so we are at liberty to use excluded middle.) Hence

$$\begin{aligned}(x,z) \in *(I_1,I_2) &\Leftrightarrow \forall y((x,y) \in I_1) \vee ((y,z) \in I_2)\\ &\Leftrightarrow \forall y(x \in U_1 \vee y \in V_1 \vee y \in V_2 \vee z \in W_2)\\ &\Leftrightarrow (x \in U_1 \vee z \in W_2) \vee Y \subseteq V_1 \cup V_2\\ &\Leftrightarrow (x,z) \in U_1 \mathrm{O} W_2 \vee Y \subseteq V_1 \cup V_2\end{aligned}$$

Say now $R_1 \rightarrowtail X \times Y, R_2 \rightarrowtail Y \times Z$ are close sublocales. Define

$$R_1; R_2 = \neg * (a_{R_1}, a_{R_2})$$

where a_{R_i} is the open corresponding to the closed sublocale R_i and $* : \Omega(X \times Y) \times \Omega(Y \times Z) \to \Omega(X \times Z)$ is defined on generators as

$$*(a_1 \mathrm{O} b_1, b_2 \mathrm{O} c_2) = a_1 \mathrm{O} c_2 \vee \Omega!(1 \leq b_1 \vee b_2)$$

where ! is the unique locale map $! : X \to 1$, and we are viewing $(1 \leq a)$ as an element of Ω.
Infact we need to factor $*$ throught $\bar{*}$:

$$\begin{aligned}\bar{*} : \Omega X \otimes \Omega Y \otimes \Omega Z &\to \Omega X \otimes \Omega Z\\ a \mathrm{O} b \mathrm{O} c &\mapsto a \mathrm{O} c \vee \Omega!(1 \leq b)\end{aligned}$$

since to make sure that we are defining a function we need to define it on all generators of some tensor. We need to check that $\bar{*}$ is well defined. i.e. that

$$(a,b,c) \mapsto a \mathrm{O} c \vee \Omega!(1 \leq b)$$

is a preframe trihomomorphism. This follows from the compactness of ΩY. Then take $*(I_1, I_2) = \bar{*}(\coprod_{12} I_1 \vee \coprod_{23} I_2)$ where the $\coprod$'s are frame coprojections.

Given a closed sublocale $\neg a \rightarrowtail X$ and some relation $\leq \rightarrowtail X \times X$ then we can find a formula for $\Downarrow \neg a$. Assuming $\leq = \neg b \mathrm{O} c$ then

$$\Downarrow \neg a = \neg(b \vee \Omega!(1 \leq c \vee a))$$

It is not immediate that even if $\leq$ is reflexive that $\neg a \leq \Downarrow \neg a$. This will be the case once we are sure that the diagonal is closed. For if $\Delta : X \rightarrowtail X \times X$ is closed then it can be checked that it is the identity with respect to relational composition. To see this recall that $\Delta = \neg \#$ where

$$\# = \bigvee\nolimits^{\uparrow}\{\wedge_i (a_i \mathrm{O} b_i) | \wedge_{i \in I} (a_i \vee b_i) = 0 \ I \text{ finite}\} \text{ (!!)}$$

But we are working with Stone locales. These are compact regular and so have closed diagonals. (Proposition III 1.3 of Stone Spaces [Joh82].)

Of course we are more familiar with the fact:

$$\# = \bigvee\{a \otimes b | a \wedge b = 0\}$$

and so one needs to translate this to its 'preframe version' (!!). In the next section we need the 'preframe version' of the ordered Stone locale condtion $a_{\leq} = \bigvee\{a \otimes \neg a | a \in K\Omega X, \Downarrow \neg a = a\}$, this is

$$a_{\leq} = \bigvee\nolimits^{\uparrow}\{\wedge_i (a_i \mathrm{O} \neg b_i) | \wedge_{i \in I} (a_i \vee \neg b_i) = 0 \quad a_i, b_i \in K\Omega X, \Downarrow \neg a_i = a_i, \Downarrow \neg b_i = \neg b_i, I \text{ finite}\}$$

5 Localic Priestly Duality

Given lemma (2.2) it is clear that the localic part of

$$\mathcal{B}: \mathsf{CohLoc} \longrightarrow \mathsf{OStoneLoc}$$

should be the Stone locale whose frame of opens is the ideal completion of the free Boolean algebra on the distributive lattice of compact opens of the domain locale. So

$$\mathcal{B}X = (\mathcal{B}X, \leq_{\mathcal{B}X})$$

where $\Omega\mathcal{B}X = IdlB_X$ where B_X is the free Boolean algebra on the distributive lattice $K\Omega X$. Note that there is a distributive lattice inclusion of $K\Omega X$ into B_X which induces a locale map $l_X : \mathcal{B}X \to X$. Since B_X is the free Boolean algebra on $K\Omega X$ we can prove that l_X is monic.
We define $\leq_{\mathcal{B}X}$ by

$$a_{(\leq_{\mathcal{B}X})} = \bigvee^{\uparrow}\{\wedge_i(a_i \mathrm{O} \neg b_i) | \wedge_{i \in I} (a_i \vee \neg b_i) = 0 \quad a_i, b_i \in K\Omega X \quad I \text{ finite}\}$$

Notice that $a_{(\leq_{\mathcal{B}X})} \leq \#$ and so

$$\Delta \leq_{\mathsf{Sub}(X \times X)} (\leq_{\mathcal{B}X}).$$

Hence $\leq_{\mathcal{B}X}$ is reflexive. Antisymmetry for $\leq_{\mathcal{B}X}$ can also be checked, but the proof is slightly more involved:

Lemma 5.1 $\leq_{\mathcal{B}X}$ is antisymmetric.

Proof: We need to prove that $(\leq_{\mathcal{B}X}) \wedge (\geq_{\mathcal{B}X}) \overset{(p_1,p_2)}{\rightarrowtail} \mathcal{B}X \times \mathcal{B}X$ is the diagonal. We may conclude this provided we check that its right hand projection is equal to its left hand projection. i.e. $p_1 = p_2$. As a statement about frames this reads

$$\Omega(\pi_1)(I) \vee a_{\leq} \vee a_{\geq} = \Omega(\pi_2)(I) \vee a_{\leq} \vee a_{\geq} \quad \forall I \in IdlB_X$$

But we only need worry about compact Is. i.e. we may assume $I = a \in B_X$. In such a case $\Omega\pi_1 I = a\mathrm{O}0$, $\Omega\pi_2 I = 0\mathrm{O}a$. Finally note that we may further restrict to the case that $a \in K\Omega X$. This is because l_X is a monomorphism.
Hence we need

$$a\mathrm{O}0 \vee a_{\leq} \vee a_{\geq} = 0\mathrm{O}a \vee a_{\leq} \vee a_{\geq} \quad \forall a \in K\Omega X$$

Before proof note that for any $a \in K\Omega X$ since $(a \vee 0) \wedge (0 \vee \neg a) = 0$ we have that

$$\begin{aligned} a_{\leq} &= a_{\leq} \vee [(a\mathrm{O}0) \wedge (0\mathrm{O}\neg a)] && (I) \\ a_{\geq} &= a_{\geq} \vee [(\neg a\mathrm{O}0) \wedge (0\mathrm{O}a)] && (II) \end{aligned}$$

Hence for any $a \in K\Omega X$

$$\begin{aligned} a\mathrm{O}0 \vee a_{\leq} \vee a_{\geq} &= a_{\leq} \vee [[a_{\geq} \vee (\neg a\mathrm{O}0) \vee (a\mathrm{O}0)] \wedge [a_{\geq} \vee (a\mathrm{O}a)]] \text{ by } (II) \\ &= a_{\leq} \vee a_{\geq} \vee (a\mathrm{O}a) \\ 0\mathrm{O}a \vee a_{\leq} \vee a_{\geq} &= a_{\geq} \vee [[a_{\leq} \vee (a\mathrm{O}a)] \wedge [a_{\leq} \vee (0\mathrm{O}\neg a) \vee (0\mathrm{O}a)]] \text{ by } (I) \\ &= a_{\leq} \vee a_{\geq} \vee (a\mathrm{O}a) \qquad \square \end{aligned}$$

Once the following lemma is checked then it is easy to see that not only will $\leq_{(\mathcal{B}X)}$ satisfy the ordered Stone locale condition but also for any coherent locale X we have $\mathcal{C}\mathcal{B}X \cong X$:

Lemma 5.2 If B_X is the free Boolean algebra on the distributive lattice of compact opens $K\Omega X$ of some coherent locale X and if $\Downarrow$ is the lower closure operation on closed sublocales induced by the order $\leq_{\mathcal{B}X}$ then for any $a \in B_X$ we have

$a \in K\Omega X$ if and only if $\Downarrow \neg a = \neg a$

Proof: Since $\leq_{\mathcal{B}X}$ is reflexive all we need to show is $\forall a \in B_X$

$$\Downarrow \neg a \leq \neg a \quad \Leftrightarrow \quad a \in K\Omega X$$

Now

$$a_{(\leq_{\mathcal{B}X})} = \bigvee^{\uparrow}\{\wedge_i(a_i O \neg b_i) | \wedge_{i\in I}(a_i \vee \neg b_i) = 0 \quad a_i, b_i \in K\Omega X \quad I \text{ finite}\}$$

and so

$$\Downarrow \neg a = \neg(\bigvee^{\uparrow}\{\wedge_i(a_i \vee \Omega!(1 \leq a \vee \neg b_i)) | \wedge_{i\in I}(a_i \vee \neg b_i) = 0, \\ a_i, b_i \in K\Omega X \; I \text{ finite}\})$$

Hence if $a \in K\Omega X$ take $I = \{1, 2\}$ and

$$a_1 = a \qquad b_1 = 1$$
$$a_2 = 0 \qquad b_2 = \neg a$$

to see that $\Downarrow \neg a \leq \neg a$.

Conversely say $\Downarrow \neg a \leq \neg a$. Then

$$a \leq \bigvee^{\uparrow} \wedge_i(a_i \vee \Omega!(1 \leq a \vee \neg b_i)$$

where the join is over finite collections of a_i, b_is in $K\Omega X$ such $\wedge_i(a_i \vee \neg b_i) = 0$. But a is a compact open and so $\exists$ a finite set I such that

$$\wedge_{i\in I}(a_i \vee \neg b_i) = 0$$

and

$$a \leq a_i \vee \Omega!(1 \leq a \vee \neg b_i) \qquad \forall i \in I$$

and the a_i, b_is are all in $K\Omega X$. However it can be seen that

$$a_i \vee \Omega!(1 \leq a \vee \neg b_i) = \bigvee^{\uparrow}(\{a_i\} \cup \{1 | 1 \leq a \vee \neg b_i\})$$

and so, using compactness of a again, we find that there are finite sets $J_1, J_2 \subseteq I$ with $I = J_1 \cup J_2$ such that $a \leq a_i \quad \forall i \in J_1$ and $1 \leq a \vee \neg b_i \quad \forall i \in J_2$. i.e.

$$a \leq \wedge_{i\in J_1} a_i \text{ and } \vee_{i\in J_2} b_i \leq a$$

The finite distributivety law allows us to prove

$$\wedge_i(a_i \vee \neg b_i) = \bigvee[(\wedge_{i\in K_1} a_i) \wedge (\wedge_{i\in K_2} \neg b_i)] \qquad (!)$$

where the join is over all pairs $K_1, K_2 \subseteq I$ such that K_1, K_2 are finite and $I = K_1 \cup K_2$. Hence

$$(\wedge_{i\in J_1} a_i) \wedge (\wedge_{i\in J_2} \neg b_i) = 0$$
$$\text{i.e.} \quad \wedge_{i\in J_1} a_i \leq \vee_{i\in J_2} b_i$$

Hence $a = \wedge_{i\in J_1} a_i \in K\Omega X$. □

Lemma 5.3 $\leq_{\mathcal{B}X}$ is transitive.

Proof: To prove transitivity of $\leq_{\mathcal{B}X}$ it is clearly sufficient to show that for any finite collection of a_i, b_is in $K\Omega X$ with $\wedge_i(a_i \vee \neg b_i) = 0$ we have

$$\wedge_i(a_i O \neg b_i) \leq \bigvee^{\uparrow} \wedge_{(\bar{i}, i)}[(\bar{a}_{\bar{i}} O \neg b_i) \vee \Omega!(1 \leq a_i \vee \neg \bar{b}_{\bar{i}})]$$

where the join is over all finite collections $(\bar{a}_{\bar{i}}, \bar{b}_{\bar{i}})_{\bar{i}\in\bar{I}}$ of elements of $K\Omega X$ such that $\wedge_{\bar{I}}(\bar{a}_{\bar{i}} \vee \neg \bar{b}_{\bar{i}}) = 0$
But $\neg a_i$ is lower closed by the last lemma since $a_i \in K\Omega X$ and so

$$a_i = \bigvee^{\uparrow} \wedge_{\bar{i}}(a_{\bar{i}} \vee \Omega!(1 \leq a_i \vee \neg \bar{b}_{\bar{i}}))$$

and since $\bigvee^\uparrow$ and finite meets commute in the theory of preframes (and O commutes with $\bigvee^\uparrow$ and finite meets in the appropriate way) we see that

$$\wedge_i(a_i \mathrm{O} \neg b_i) = \bigvee^\uparrow \wedge_{(i,\bar{i})}((\bar{a}_{\bar{i}} \mathrm{O} \neg b_i) \vee \Omega!(1 \leq a_i \vee \neg \bar{b}_{\bar{i}})) \qquad \square$$

So finally all we do is check that $\mathcal{BC}(Y) \cong Y$ for all $Y \in$ OStoneLoc. We know that there is a distributive lattice inclusion,

$$\{a \in K\Omega Y | \Downarrow \neg a = \neg a\} \hookrightarrow K\Omega Y$$

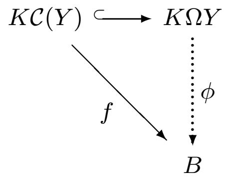
but is it universal? If we can show this then the fact that for any ordered Stone locale $(Y, \leq_Y)$ we have

$$a_{\leq_Y} = \bigvee^\uparrow \{\wedge(a_i \mathrm{O} \neg b_i) | \wedge_i (a_i \vee \neg b_i) = 0, \quad a_i, b_i \in K\Omega Y \quad \Downarrow \neg a_i = \neg a_i, \Downarrow \neg b_i = \neg b_i\}$$

allows us to conclude

$$\leq_Y = \leq_{\mathcal{BC}(Y)} .$$

Thus we will be finished provided we can check the universality of the above inclusion. Assume a diagram



where f is a distributive lattice homomorphism and B is a Boolean algebra. Say $a \in K\Omega Y$ and we have found two finite sets of elements $\{a_i, b_i | i \in I\}$, $\{\bar{a}_{\bar{i}}, \bar{b}_{\bar{i}} | \bar{i} \in \bar{I}\}$ such that $\wedge_i(a_i \vee \neg b_i) = a = \wedge_{\bar{i}}(\bar{a}_{\bar{i}} \vee \neg \bar{b}_{\bar{i}})$. (Where the $a_i, b_i, \bar{a}_{\bar{i}}, \bar{b}_{\bar{i}}$'s are in $\{a \in K\Omega Y | \Downarrow \neg a = \neg a\}$). We want to check,

Lemma 5.4 $\wedge_i(fa_i \vee \neg fb_i) = \wedge_{\bar{i}}(f\bar{a}_{\bar{i}} \vee \neg f\bar{b}_{\bar{i}})$

(For then it will be 'safe' to define $\phi(a) = \wedge_i(fa_i \vee \neg fb_i)$ for any $\{a_i, b_i | i \in I\} \subseteq K\mathcal{C}(Y)$ such that $a = \wedge_i(a_i \vee \neg b_i)$)
Proof: To conclude that $\wedge_i(fa_i \vee \neg fb_i) \leq \wedge_{\bar{i}}(f\bar{a}_{\bar{i}} \vee \neg f\bar{b}_{\bar{i}})$ we need to prove that for every $\bar{i}$ and for every pair $J_1, J_2 \subseteq I$ with $I \subseteq J_1 \cup J_2$ we have

$$(\wedge_{i \in J_1} fa_i) \wedge (\wedge_{i \in J_2} \neg fb_i) \leq (f\bar{a}_{\bar{i}} \vee \neg f\bar{b}_{\bar{i}})$$

To see this apply the finite distributivety law (!) of lemma 5.2 to the meet $\wedge_i(fa_i \vee \neg fb_i)$. But the last inequality can be manipulated to

$$f((\wedge_{i \in J_1} a_i \wedge \bar{b}_{\bar{i}}) \vee \vee_{i \in J_2} b_i) \leq f((\bar{a}_{\bar{i}} \wedge \bar{b}_{\bar{i}}) \vee (\vee_{i \in J_2} b_i))$$

and the fact that $(\wedge_{i \in J_1} a_i \wedge \bar{b}_{\bar{i}}) \vee \vee_{i \in J_2} b_i \leq (\bar{a}_{\bar{i}} \wedge \bar{b}_{\bar{i}}) \vee (\vee_{i \in J_2} b_i)$ follows from exactly the same manipulations applied to the assumption

$$\wedge_i(a_i \vee \neg b_i) \leq \wedge_{\bar{i}}(\bar{a}_{\bar{i}} \vee \neg \bar{b}_{\bar{i}}) \ \square$$

It also follows (given the assumption that $\forall a \in K\Omega Y \quad \exists \{a_i, b_i | i \in I\} \subseteq K\mathcal{C}Y$ s.t. $\wedge_i(a_i \vee \neg b_i) = a$) that ϕ will be a (necessarily unique) Boolean homomorphism. [For if $a = \wedge_{i \in I}(a_i \vee \neg b_i)$ and $\bar{a} = \wedge_{i \in \bar{I}}(a_i \vee \neg b_i) \quad \Rightarrow a \wedge \bar{a} = \wedge_{I \cup \bar{I}}(a_i \vee \neg b_i)$. So

$$\begin{aligned} \phi(a \wedge \bar{a}) &= \wedge_{I \cup \bar{I}}(fa_i \vee \neg fb_i) \\ &= [\wedge_{i \in I}(fa_i \vee \neg fb_i)] \wedge [\wedge_{i \in \bar{I}}(fa_i \vee \neg fb_i)] \\ &= \phi(a) \wedge \phi(\bar{a}) \end{aligned}$$

Similarly for $\vee$]
We also have the following Boolean algebra lemma:

Lemma 5.5 If $I, \bar{I}$ are finite sets and $\{a_i, b_i | i \in I\}$ and $\{\bar{a}_{\bar{i}}, \bar{b}_{\bar{i}} | \bar{i} \in \bar{I}\}$ are sets of elements of some Boolean algebra B, and $\wedge_i (a_i \vee \neg b_i) = 0, \wedge_{\bar{i}} (\bar{a}_{\bar{i}} \vee \neg \bar{b}_{\bar{i}}) = 0$. Then for any $J_1, J_2 \subseteq I \times \bar{I}$ such that $I \times \bar{I} \subseteq J_1 \cup J_2$ we have
$\wedge_{(i,\bar{i}) \in J_1} (a_i \vee \neg \bar{b}_{\bar{i}}) \leq \vee_{(i,\bar{i}) \in J_2} (\neg \bar{a}_{\bar{i}} \wedge b_i)$

Proof: The conditions imply:

$$\begin{aligned} & [\wedge (a_i \vee \neg b_i)] \vee [\wedge (\bar{a}_{\bar{i}} \vee \neg \bar{b}_{\bar{i}})] = 0 \\ \Rightarrow \quad & \wedge_{(i,\bar{i}) \in I \times \bar{I}} [a_i \vee \neg b_i \vee \bar{a}_{\bar{i}} \vee \neg \bar{b}_{\bar{i}}] = 0 \\ \Rightarrow \quad & \vee_{I \times \bar{I} \subseteq J_1 \cup J_2} [(\wedge_{(i,\bar{i}) \in J_1} (a_i \vee \neg \bar{b}_{\bar{i}})) \wedge (\wedge_{(i,\bar{i}) \in J_2} (\bar{a}_{\bar{i}} \vee \neg b_i))] = 0 \\ \Rightarrow \quad & (\wedge_{(i,\bar{i}) \in J_1} (a_i \vee \neg \bar{b}_{\bar{i}})) \wedge (\wedge_{(i,\bar{i}) \in J_2} (\bar{a}_{\bar{i}} \vee \neg b_i)) = 0 \end{aligned}$$

The result follows since

$$\neg (\wedge (\bar{a}_{\bar{i}} \vee \neg b_i)) = \vee (\neg \bar{a}_{\bar{i}} \wedge b_i)$$

□

We can now prove our assumption:

Theorem 5.1 If $(Y, \leq)$ is an ordered Stone locale and $a \in K\Omega Y$ then $a = \wedge_{i \in I} (a_i \vee \neg b_i)$ for some finite I with $a_i, b_i \in K\Omega Y$ and $\Downarrow \neg a_i = \neg a_i$, $\Downarrow \neg b_i = \neg b_i$.

Proof: Clearly the anti-symmetry axiom must now come into play. This axiom states that

$$(\leq) \wedge (\geq) \leq_{\mathrm{Sub}(X \times X)} \Delta$$

which as a statment about opens of $\Omega(X \times X)$ reads:

$$a_{\leq} \vee a_{\geq} \geq \#$$

But $a = *(\#, a)$ since $\neg \#$ is the identity for relational composition. Thus

$$a \leq (a_{\leq} \vee a_{\geq}) * a \qquad \text{(I)}$$

From our axioms used to define 'ordered Stone locale' we know,

$$a_{\leq} = \vee^{\uparrow} \{\wedge_i (a_i \mathrm{O} \neg b_i) | \wedge_i (a_i \vee \neg b_i) = 0 \quad a_i, b_i \in K\Omega Y \quad \Downarrow \neg a_i = \neg a_i \;\; \Downarrow \neg b_i = \neg b_i\}$$

Symmetrically

$$a_{\geq} = \vee^{\uparrow} \{\wedge_{\bar{i}} (\neg \bar{b}_{\bar{i}} \mathrm{O} \bar{a}_{\bar{i}}) | \wedge_{\bar{i}} (\bar{a}_{\bar{i}} \vee \neg \bar{b}_{\bar{i}}) = 0 \quad \bar{a}_{\bar{i}}, \bar{b}_{\bar{i}} \in K\Omega Y \quad \Downarrow \neg \bar{a}_{\bar{i}} = \neg \bar{a}_{\bar{i}} \;\; \Downarrow \neg \bar{b}_{\bar{i}} = \neg \bar{b}_{\bar{i}}\}$$

Thus $a_{\leq} \vee a_{\geq}$ is a directed union of elements of the form

$$\begin{aligned} & [\wedge_i (a_i \mathrm{O} \neg b_i)] \vee [\wedge_{\bar{i}} (\neg \bar{b}_{\bar{i}} \mathrm{O} \bar{a}_{\bar{i}})] \\ = \quad & \wedge_{(i,\bar{i}) \in I \times \bar{I}} [(a_i \mathrm{O} \neg b_i) \vee (\neg \bar{b}_{\bar{i}} \mathrm{O} \bar{a}_{\bar{i}})] \\ = \quad & \wedge_{(i,\bar{i}) \in I \times \bar{I}} [(a_i \vee \neg \bar{b}_{\bar{i}}) \mathrm{O} (\neg b_i \vee \bar{a}_{\bar{i}})] \end{aligned}$$

Since a is compact and $(\_) * a$ preserves directed joins and finite meets we see from (I) that

$$a \leq \wedge_{(i,\bar{i}) \in I \times \bar{I}} ([(a_i \vee \neg \bar{b}_{\bar{i}}) \mathrm{O} (\neg b_i \vee \bar{a}_{\bar{i}})] * a)$$

for some $\{a_i, b_i | i \in I\}, \{\bar{a}_{\bar{i}}, \bar{b}_{\bar{i}} | \bar{i} \in \bar{I}\}$ such that $\wedge_i(a_i \vee \neg b_i) = 0, \wedge_{\bar{i}}(\neg \bar{b}_{\bar{i}} \vee \bar{a}_{\bar{i}}) = 0$ and $\Downarrow \neg a_i = \neg a_i, \Downarrow \neg b_i = \neg b_i, \Downarrow \neg \bar{a}_{\bar{i}} = \neg \bar{a}_{\bar{i}}, \Downarrow \neg \bar{b}_{\bar{i}} = \neg \bar{b}_{\bar{i}}$. Now

$$\begin{array}{cl} & [(a_i \vee \neg \bar{b}_{\bar{i}}) \mathrm{O} (\neg b_i \vee \bar{a}_{\bar{i}})] * a \\ = & (a_i \vee \neg \bar{b}_{\bar{i}}) \vee \Omega!(1 \leq \neg b_i \vee \bar{a}_{\bar{i}} \vee a) \\ = & \vee^{\uparrow}[\{a_i \vee \neg \bar{b}_{\bar{i}}\} \cup \{1 | b_i \wedge \neg \bar{a}_{\bar{i}} \leq a\}] \end{array}$$

We see from the compactness of a that there are two finite sets $J_1, J_2, \subseteq I \times \bar{I}$ such that

$$I \times \bar{I} \subseteq J_1 \cup J_2$$

and

$$a \leq \wedge_{(i,\bar{i}) \in J_1}(a_i \vee \neg \bar{b}_{\bar{i}})$$
$$\vee_{(i,\bar{i}) \in J_2}(\neg \bar{a}_{\bar{i}} \wedge b_i) \leq a$$

But by the last lemma

$$\wedge_{(i,\bar{i}) \in J_1}(a_i \vee \neg \bar{b}_{\bar{i}}) \leq \vee_{(i,\bar{i}) \in J_2}(\neg \bar{a}_{\bar{i}} \wedge b_i)$$

and so $a = \wedge_{(i,\bar{i}) \in J_1}(a_i \vee \neg \bar{b}_{\bar{i}})$ □

6 Morphisms and final remarks

Clearly some spatial intuitions have been lost in this exposition in an attempt to prove the result as quickly as possible. Foremost we have not given any justification for the choice of $\leq_{\mathcal{B}X}$ other than “it works”. Infact $\leq_{\mathcal{B}X}$ is the pullback of the specialization order along $l_X \times l_X$. Antisymmetry of $\leq_{\mathcal{B}X}$ thn follows immediately since l_X is monic and meets are pullback stable.

We say $f : (X \leq_X) \to (Y, \leq_Y)$ is an ordered Stone locale map iff it “preserves order”. i.e. if and only if $\exists n : \leq_X \to \leq_Y$ such that

$$\begin{array}{ccc} \leq_X & \xrightarrow{n} & \leq_Y \\ \cap\downarrow & & \cap\downarrow \\ X \times X & \xrightarrow{f \times f} & Y \times Y \end{array}$$

commutes.
However for this definition to fit in with the algebra of the paper we need to translate it. We find that $f : (X \leq_X) \to (Y, \leq_Y)$ is an ordered Stone locale map if and only if (it is a locale map from X to Y and)

$$\Omega f \Downarrow^{op} (a) \leq \Downarrow^{op} \Omega f(a) \qquad \forall a \in \Omega Y$$

where $\Downarrow^{op}$ is lower closure viewed as an operation on the corresponding opens. i.e. $\Downarrow \neg a = \neg \Downarrow^{op} a \quad \forall a$.

For further information about this work consult [Tow96].

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