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Presenting locale pullback via directed complete posets

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Abstract

This paper shows how to describe the pullbacks of directed complete posets (dcpos) along geometric morphisms. This extends Joyal and Tierney's original results on the pullbacks of suplattices. It is then shown how to treat every frame as a dcpo and so locale pullback is described in this way. Applications are given describing triquotient assignments in terms of internal dcpo maps, leading to pullback stability results for triquotient maps. The main application here shows how dcpo maps between frames can be described in terms of certain external natural transformations.

1 Introduction

The direct image part of a geometric morphism preserves suplattices (that is, complete posets) and so defines a functor from suplattices, internal in one topos to suplattices internal to the codomain topos. Joyal and Tierney in [JoyTie 84] show that this functor has a left adjoint. This seemingly highly technical observation has important implications since it specializes to frames (complete Heyting algebras) and so provides a description of the pullback of locales along a geometric morphism. (Recall that pullback can be described as a right adjoint and that the category of locales is opposite to frames.)

The trick of this result is to use suplattice presentations. The presentations (as formal objects) are stable under the inverse image of geometric morphisms and so this defines a functor in the opposite direction to the direct image functor. That this is left adjoint amounts to checking that under the bijection defined by the adjunction of the geometric morphism, maps which satisfy R correspond to maps which satisfy f^*R where R is the set of relations in the presentation and f is the geometric morphism. So the case where R is empty is immediate since the power set on a set of generators forms the free suplattice (and $f^*\phi = \phi$).

The main objective of this paper is to extend this result to the directed complete partial orders (posets with joins for all directed subsets). The same trick is used to prove this and so the first bit of work needed is to verify that dcpo presentations present (i.e. are well defined). This result appears to be folklore and is re-proved here in an entirely constructive manner (and no natural numbers object is used). The left adjoint again describes pullback of locales along a geometric morphism since we are able to describe frames as particular objects in the category of dcpos. The way that this is done is not in keeping with Joyal and Tierney's view of frames as types of rings over suplattice tensor. Here the novel view is taken that frames are internal distributive lattices in the ordered enriched category of dcpos. The internality required is a strong one in that the join and meet operations of the internal distributive lattices are required to be consistent with the given order enrichment.

As an application of the existence of the left adjoint we have a number of known results: e.g. open and proper maps are pullback stable. This is shown by looking at locale maps with triquotient assignments (types of dcpo homomorphisms); these locale maps generalize both proper and open maps. A new result is shown describing the triquotient assignments in terms of dcpo homomorphisms internally in the topos of sheaves over the codomain locale.

The main application focused on here is the following result which is an extension of a recent result of Townsend and Vickers [TowVic 02]. It is shown that the dcpos maps between frames are exactly the natural transformations between certain functors indexed by geometric morphisms. The functors are

$$\begin{aligned} &\Lambda^{\Omega_{\mathcal{E}}W} : (\mathbf{Top}/\mathcal{E})^{op} \to SET \\ &\quad (h : \mathcal{E}' \to \mathcal{E}) \longmapsto \mathbf{Top}(\mathcal{E}' \times_{\mathcal{E}} Sh(W), Sh(\mathbb{S})) \end{aligned}$$

for any frame $\Omega_{\mathcal{E}}W$, corresponding to a locale W in $\mathcal{E}$. Here $\mathbb{S}$ is the Sierpiński locale and SET is some background category of possibly large sets. The importance of the Townsend/Vickers result appears to be that it offers insight into the parallel between proper and open in locale theory (e.g. [Townsend 96]) and so extending it to geometric morphisms may offer insight into the parallel between proper and open in topos theory. This would be the subject of further work based on the results presented here.

Along the way an exposition on topos theory has, in effect, been included. While all the results are known to those working in the field, it is hoped that the exposition offers insight into exactly how various lattice structures are translated between toposes.

2 Dcpo presentations present

For detailed background information on the lattice structures under discussion consult [Johnstone 82]. Firstly we will recall some basic definitions and notation. The category **dcpo** has as objects directed complete partial orders and has as morphisms directed join preserving maps. The category **sup** of suplattices has as objects complete posets and has as morphisms maps preserving all joins. (To prove that dcpo presentations present we shall use the fact that suplattice presentations present.)

Definition 2.1 A *dcpo presentation* is the following data: (i) a poset G (ii) a set R (which is an indexing set for relations) (iii) a function $\lambda : R \to G$ and (iv) a subset $\pi \subseteq G \times R$ such that for every $r \in R$, $\{g \mid g\pi r\}$ is a lower closed directed subset of G.

The dcpo being presented by a dcpo presentation (which shall be denoted $\mathbf{dcpo}\langle G \text{ qua poset} | \ R\rangle$) is that which universally satisfies,

$$\lambda(r) = \bigvee^{\uparrow}\{g \in G \mid g\pi r\}.$$

and preserves the order on G. The "qua" notation indicates that whatever follows the qua must be true in the object being presented, see e.g. [JoVic 91].

Example 2.2 Any dcpo A has a presentation given by (i) A, (ii) $idl(A)$, (iii) $\vee^{\uparrow} : idl(A) \to A$ and (iv) $a\pi I$ iff $a \in I$. Recall that $idl(A)$ is the set of ideals of A (an ideal of a poset is any lower closed and directed subset). It is routine to check that $A \cong \mathbf{dcpo}\langle A \textit{ qua poset} \mid idl(A)\rangle$ if the latter is well defined.

The next theorem forms a foundation to the ideas in this paper as it shows that the presentation work. The fact that they work appears to be folklore, though see [Markowsky 77] for some relevant early work.

Theorem 2.3 *(dcpo presentations present) For any dcpo presentation* $(G, R, ...)$, $\mathbf{dcpo}\langle G \textit{ qua poset} \mid R\rangle$ *is well defined.*

Proof. This proof is a reapplication of the techniques of [JoVic 91], where preframe presentations are proved to exist from the existence of frame presentations. Here, we replace the category of preframes with **dcpo** and the category of frames with **sup**. First note that the problem reduces to a proof of the existence of **dcpo** coequalizers since the ideal completion of any poset is the free dcpo on that poset. So $\mathbf{dcpo}\langle G \textit{ qua poset} \mid R\rangle$ (if defined) is the coequalizer of

$$idl(R) \underset{e_2}{\overset{e_1}{\rightrightarrows}} idl(G)$$

where $idl(R) \cong R$ (since R is a discrete poset) and $e_1(\_) = \downarrow \circ \lambda$ and $e_2(r)$ is the ideal $\{g \mid g\pi r\}$ for every $r \in R$.

Now, suplattice presentations (coequalizers) certainly exist ([JoyTie 84]; for a set of relations R on a suplattice M the set of R-coherent elements forms the coequalizer, where an $m \in M$ is R-coherent iff for every aRb it is the case that $a \leq m$ iff $b \leq m$). The key observations needed to complete the proof are that (i) **dcpo** has image factorizations and (ii) the universal mapping of a **dcpo** to its free suplattice (qua dcpo) is monic.

The category of dcpos does have image factorizations: take the least sub-dcpo generated by the set theoretic image of the function to be factorized.

Checking (ii), that the unit is monic, requires the observations that F : **dcpo** $\to$ **sup** has a concrete description: $F(A)$ is the set of Scott closed subsets of A, that is the lower closed subsets that are closed under directed joins. Any intersection of Scott closed subsets is clearly Scott closed and so $F(A)$ is certainly a suplattice. $\downarrow: A \to F(A)$ is Scott continuous (preserves directed joins), and this map will prove to be the monic unit. To see this first note that for any $B \in F(A)$, $B = \bigvee\{\downarrow b \mid b \in B\}$ since the join always contains the set theoretic union. So, given any dcpo map $\phi : A \to M$ with M a suplattice, the assignment $q : B \longmapsto \bigvee_M \{\phi(b) \mid b \in B\}$ is therefore necessary if ϕ is to factor via $\downarrow$. But $r : M \to F(A)$ given by $m \longmapsto \{b \mid \phi(b) \leq m\}$ provides a right adjoint to q so we know that q is a suplattice homomorphism, and therefore $F(A)$ defined as the set of Scott closed subsets provides the correct universal properties.

To find the dcpo coequalizer of $f, g : A \rightrightarrows B$, the first step is to take the suplattice coequalizer of Ff, Fg, giving a suplattice homomorphism $h' : F(B) \to C'$. Here F : **dcpo** $\to$ **sup** is the free functor (left adjoint to the forgetful functor). Applying the forgetful functor and precomposing with the unit, we get a dcpo morphism $h' \circ \downarrow: B \to C'$. Next take the image factorization in **dcpo** to get $i \circ h : B \to C \to C'$.

h is the required dcpo coequalizer of f and g. If $k : B \to D$ composes equally with f and g, then $F(k)$ factors via C' as $k' \circ h'$ (say). Because h is a cover and $\downarrow_D$ is monic, we get that k factors via h. (The pullback of D along $k' \circ i$ must be the whole of C.)

$$\begin{array}{ccc} B & \overset{h}{\twoheadrightarrow} & C \\ & & \downarrow i \\ k \downarrow & \swarrow & C' \\ & & \downarrow k' \\ D & \underset{\downarrow}{\hookrightarrow} & F(D) \end{array}$$

Uniqueness follows since h is an epimorphism (covers are epimorphism as dcpo has equalizers). $\Box$

This theorem also appears in [TowVic 02]. A corollary is that dcpo tensor can be defined; though before that is proved it must be made clear that:

Proposition 2.4 *Binary tensor and binary product are equivalent in the category* **dcpo**.

Proof. If A and B are dcpos then the poset $A \times B \equiv \{(a,b) \mid a \in A, b \in B\}$ is a dcpo since if $I \subseteq^{\uparrow} A \times B$ then $\bigvee^{\uparrow} I = (\bigvee^{\uparrow} \pi_1[I], \bigvee^{\uparrow} \pi_2[I])$. $A \times B$ can be verified to be dcpo product in the usual manner. To show that it is also dcpo tensor it must be verified that bi-dcpo-linear maps $A \times B \to C$ are the same thing as dcpo maps $A \times B \to C$. This is trivial from the definition of join in $A \times B$ just given (and the definition of directed). $\square$

Whilst we have a very simple description of tensor (it is the same as set-theoretic product) it will turn out that its more complicated description in terms of the presentation of the tensor will be what is needed to show that frames can be treated as internal distributive lattices (in dcpo) when moving from one topos to another.

Proposition 2.5 *If $(G_1, R_1, ...)$ and $(G_2, R_2, ...)$ are two dcpo presentations then $(G_1 \times G_2, G_1 \times R_2 \amalg R_1 \times G_2, ...)$ presents their tensor where $\lambda_\otimes : G_1 \times R_2 \amalg R_1 \times G_2 \to G_1 \times G_2$ is given by $\lambda_\otimes = [\lambda_L, \lambda_R]$ where $\lambda_L(g_1, r_2) = (g_1, \lambda_2(r_2))$ and $\lambda_R(r_1, g_2) = (\lambda_1(r_1), g_2)$, and $\pi_\otimes \subseteq (G_1 \times G_2) \times (\ G_1 \times R_2 \amalg R_1 \times G_2) \cong (G_1 \times G_2) \times (\ G_1 \times R_2) \amalg (G_1 \times G_2) \times (\ R_1 \times G_2)$ is given $\pi_\otimes = \pi_L \amalg \pi_R$ with $\pi_L \subseteq (G_1 \times G_2) \times (G_1 \times R_2)$ given by $(g_1', g_2)\pi_L(g_1, r_2)$ iff $g_1' \leq g_1$ and $g_2 \pi_2 r_2$, and $\pi_R \subseteq (G_1 \times G_2) \times (\ R_1 \times G_2)$ given by $(g_1, g_2')\pi_L(r_1, g_2)$ iff $g_2' \leq g_2$ and $g_1 \pi_1 r_1$.*

Proof. This presentation is a re-expression of the definition of tensor. It must be verified that monotone maps $\phi : G_1 \times G_2 \to A$ (for any dcpo A) which satisfy $R_\otimes \equiv G_1 \times R_2 \amalg R_1 \times G_2$ correspond to exactly the bi-dcpo-linear maps $A_1 \times A_2 \to A$ where A_i is presented by $(G_i, R_i, ...)$ for $i = 1, 2$. Now, the category of dcpos is cartesian closed (function space directed join is calculated pointwise) and so given such a ϕ, its exponential transpose $G_2 \to A^{G_1}$ satisfies R_2, since

$$
\begin{aligned}
\phi(g_1, \lambda_2(r_2)) &= \bigvee\nolimits^{\uparrow}_{A} \{\phi(g_1', g_2) \mid g_1' \leq g_1,\ g_2 \pi_2 r_2\} \\
&= \bigvee\nolimits^{\uparrow}_{A} \{\phi(g_1, g_2) \mid \ g_2 \pi_2 r_2\}
\end{aligned}
$$

and so $\phi(\_, \lambda_2(r_2)) = \bigvee^{\uparrow}_{A^{G_1}} \{\phi(\_, g_2) \mid g_2 \pi_2 r_2\}$. Similarly with the Right equations given by π_R and A_2 in the place of G_1. Therefore any such ϕ gives rise to a bi-dcpo-linear map $A_1 \times A_2 \to A$. The same argument can be seen to work in reverse, i.e. bi-dcpo-linear maps give rise to $\phi : G_1 \times G_2 \to A$ which satisfy $R_\otimes$. But, by the definition of universal dcpo presentation, and the definition of exponentiation, this correspondence is a bijection. $\square$

3 Internal dcpos

In this section some basic definitions and lemmas about internal posets in a topos are recalled. For any topos, $\mathcal{E}$, an internal preorder is an internal category $\leq \underset{p_2}{\overset{p_1}{\rightrightarrows}} G$ such that $\leq \overset{(p_1,p_2)}{\to} G \times G$ is monic. A internal poset is an internal preorder such that G is the pullback of $\leq$ along $\geq$. A monotone map between posets is a an internal functor.

Lemma 3.1 *For any internal poset G in $\mathcal{E}$, and any object I of $\mathcal{E}$ the homset $\mathcal{E}(I,G)$ is an external poset, with $f \leq g : I \to G$ iff $I \stackrel{(f,g)}{\to} G \times G$ factors through $\leq_G$.*

This lemma is immediate from the definition of internal poset. Note that we have used "homset" but more accurately it should be "homclass" and, perhaps, poclass (rather than poset). Externally there is a big category SET of all sets and classes, in which the homsets of all toposes live. An adjunction between arbitrary categories, for example, is a class indexed collection of bijections between homclasses. It is part of the external structure that is simply taken for granted in what follows. SET will not be discussed, and so the term *set* will mean "object in the topos under consideration", e.g. an object of $\mathcal{E}$.

Here are the standard definitions now written out for object (sets) internal to $\mathcal{E}$.

Definition 3.2 (i) An internal poset, I, is *directed* iff (a) the map $I \stackrel{!}{\to} 1$ is a surjection (i.e. regular epi, i.e. I non-empty) and (b) the map $\pi_{13} : \leq \times_G \geq \to G \times G$, (i.e. $\{(i,k,j) \mid i \leq k, j \leq k\} \to G \times G$ given by $(i,k,j) \longmapsto (i,j)$) is a regular epimorphism.

(ii) For any internal posets, I and G, the external poset of all monotone maps from I to G is denoted $\mathbf{Pos}_{\mathcal{E}}(I,G)$.

(iii) For any internal poset A the subposet of the internal poset PA consisting of the lower closed directed subsets of A (i.e. the ideals of A) is denoted $idl(A)$. There is an inclusion $\downarrow \in \mathbf{Pos}_{\mathcal{E}}(A, idl(A))$.

(iv) Given two internal posets A, B and elements $f \in \mathbf{Pos}_{\mathcal{E}}(A,B)$, $g \in \mathbf{Pos}_{\mathcal{E}}(B,A)$ then f is left adjoint to g iff $1_A \leq g \circ f$ and $f \circ g \leq 1_B$ in the external orders of $\mathbf{Pos}_{\mathcal{E}}(A,A)$, $\mathbf{Pos}_{\mathcal{E}}(B,B)$ respectively.

(v) M (a poset in $\mathcal{E}$) is a suplattice iff there exists an internal functor $\bigvee : PM \to M$ which is left adjoint to $\downarrow : M \to PM$.

(vi) A (a poset in $\mathcal{E}$) is a dcpo iff there exists an internal functor $\bigvee^{\uparrow} : idlA \to A$ which is left adjoint to $\downarrow : A \to idlA$.

Suplattices and dcpos are therefore defined by reference to the existence of internal maps. That these correspond to the usual external definitions (in terms of being cocomplete/filtered cocomplete as a category) will be the next objective. This is well known topos theory, at least for suplattices, see e.g. B 2.3.9 in [Johnstone 02]. We will need the definition of a fiber directed map to help formulate the external notion of a dcpo:

Definition 3.3 An internal monotone map $x : I \to J$ is *fiber directed* iff

(a) for every $j \in J$ the set $\{i \mid x(i) = j\}$ is directed and

(b) for every $j_1 \leq j_2$ in J, $\{i \mid x(i) = j_1\} \subseteq \downarrow \{i \mid x(i) = j_2\}$

Perhaps the expression should be "fiber directed and closed with respect to the orders", but the expression "fiber directed" will be used. For example, every directed poset I, $! : I \to 1$ is fiber directed, and it is easy to verify that the pullback of a fiber directed map along a monotone map is fiber directed. Using the perhaps more familiar notion of an approximable map (e.g. [Scott 82] or [Vickers 93]), we have that a monotone map is fiber directed if and only if the left lower closure of its graph (i.e. $\{(i,j) \in I \times J \mid \exists i' \in I, i \leq i', x(i') = j\}$) is an approximable mapping. Recall that, given any two posets A, B an *approximable mapping* is a subset $R \subseteq A \times B$ such that

(*a*) $\{a \mid aRb\}$ is lower closed and directed for every $b \in B$ and
(*b*) $\forall b_1 \leq b_2 \in B, \ \{a \mid aRb_1\} \subseteq \{a \mid aRb_2\}$.

It is worth noting that the approximable maps have a natural place in a topos since $idl(A)$ classifies them:

Proposition 3.4 *The membership relation* $\in_A \to A \times idlA$ *classifies approximable mappings. I.e. there is an order isomorphism* $\mathbf{AMap}_{\mathcal{E}}(A \times B) \cong \mathbf{Pos}_{\mathcal{E}}(B, idlA)$ *given by pullback of* $\in_A$.

Proof. This is immediate from the fact that PA classifies relations (i.e. $Sub(A \times B) \cong \mathcal{E}(B, PA)$ via pullback of $\in_A$) and unravelling the definitions. (Simply argue as if this is the category of sets.) □

The external definitions can now be given and shown to coincide with the internal definitions:

Lemma 3.5 *(i) Given M an internal poset, M is a suplattice iff $x^* : \mathcal{E}(J, M) \to \mathcal{E}(I, M)$ has a left adjoint Σ_x for every $x : I \to J$ and for any pullback square*

$$\begin{array}{ccc} I \times_J K & \overset{p_2}{\to} & K \\ p_1 \downarrow & & \downarrow y \\ I & \overset{x}{\to} & J \end{array}$$

in $\mathcal{E}$, $x^*\Sigma_y = \Sigma_{p_1} p_2^*$ (i.e. the Beck-Chevalley condition holds).

(ii) Given A an internal poset, A is a dcpo iff $x^* : \mathbf{Pos}_{\mathcal{E}}(J, A) \to \mathbf{Pos}_{\mathcal{E}}(I, A)$ has a left adjoint Σ_x for every fiber directed monotone $x : I \to J$ and the Beck-Chevalley condition holds (for pullbacks of fiber directed maps along monotone maps).

The standard notation is used that if $x : I \to J$ is a morphism in $\mathcal{E}$ and G is an internal poset then $x^* : \mathcal{E}(J, G) \to \mathcal{E}(I, G)$, is defined as "precompose with x". It can be verified that it is always a monotone map. Its left adjoint, when it exists, is denoted Σ_x.

Proof. (i) If M is a suplattice then given $x : I \to J$ and $k : I \to M$ define $\Sigma_x(k)(j) = \bigvee\{k(i) \mid x(i) = j\}$ where the join is defined since M is a suplattice.

But for any $z : J \to M$,

$$(\forall j)\ \bigvee_{x(i)=j} k(i) \le z(j) \text{ if and only if } \forall i,\ k(i) \le zx(i)$$

and so left adjoints have been defined. For the Beck-Chevalley condition, say $l : K \to M$. Then $[x^*\Sigma_y(l)](i) = \Sigma_y(l)x(i) = \bigvee\{l(k) \mid y(k) = x(i)\}$. But $[\Sigma_{p_1}p_2^*(l)](i) = \Sigma_{p_1}(lp_2)(i) = \bigvee\{lp_2(i',k') \mid p_1(i',k') = i\} = \bigvee\{l(k') \mid y(k') = x(i)\}$ where the last line is because $(i',k') \in I \times_J K$ iff $y(k') = x(i')$.

Conversely, say $\in_M \stackrel{(n,e)}{\to} PM \times M$ is the membership relation on M. Then $\bigvee$ can be defined as $\Sigma_n(e)$. The proof is in B2.3.9 in [Johnstone 02], or adapt the proof of (ii) to follow.

(ii) Firstly say that A is a dcpo and we are given $x : I \to J$, fiber directed. Then for any $k : I \to A$ (a monotone map to A) and for any $j \in J$, $\{k(i) \mid x(i) = j\}$ is directed. So the constructions of (i) are available, in particular note that $\Sigma_x(k)(j)$ is monotone in j by the (b) part of the definition of fiber directed.

Conversely let $\in_A \stackrel{(n,e)}{\to} idlA \times A$ be the membership relation on A (we are using the opposite relation for convenience, this is just notation). $\in_A$ is a subposet $idlA \times A$ and $n : \in_A \to idl(A)$ is a fiber directed monotone map by definition of ideal. Define $\bigvee^{\uparrow}$ as $\Sigma_n(e)$. The pullback of n along $\downarrow$ is $\pi_2 : \le \to A$, and let $z : \le \to \in_A$ be the top arrow of this pullback. Then $\pi_1 : \le \to A = ez$. By Beck-Chevalley on this pullback square

$$\bigvee^{\uparrow} \downarrow = \downarrow^* [\Sigma_n(e)] = [\downarrow^* \Sigma_n](e) = [\Sigma_{\pi_2} z^*](e) = \Sigma_{\pi_2}\pi_1$$

and so $\bigvee^{\uparrow} \downarrow \le 1$ since $\pi_1 \le \pi_2$ and Σ_{π_2} is left adjoint to π_2^*.

To complete the proof apply the preceding proposition to the approximable maps classified by $\downarrow \bigvee^{\uparrow}$ and 1_{idlA}. Since $\bigvee^{\uparrow} n = n^*\Sigma_n(e) \ge e$, i.e. $(e, \bigvee^{\uparrow} n) : \in_A \to A \times A$ factors through $\le \to A \times A$ (recall Lemma 3.1). Then it can be verified that $\in_A \stackrel{(n,e)}{\to} idl(A) \times A \stackrel{\downarrow\bigvee^{\uparrow}\times 1}{\to} idl(A) \times A$ factors through $\in_A \stackrel{(n,e)}{\to} idlA \times A$ (use $z : \le \to \in_A$, the pullback of $\downarrow$ along n), and so the result follows from the proposition since $\in_A \stackrel{(n,e)}{\to} idlA \times A$ classifies the identity. □

Thus the usual internal definition of a dcpo in terms of having a map $\bigvee^{\uparrow} : idl(A) \to A$ corresponds to the external definition that will be used here. Finally for this section we verify that the internal definition of dcpo homomorphism (i.e. a map $f : A \to B$ such that $f(\bigvee^{\uparrow}_A I) = \bigvee^{\uparrow}_B \{f(a) \mid a \in I\}$ for every I in $idl(A)$), corresponds to the external definition of being a directed join preserving internal functor.

Definition 3.6 $f : A \to B$, a monotone map between dcpos, is *externally a dcpo map* iff for any fiber directed $x : I \to J$, $f \circ [\Sigma^A_x(k)] = \Sigma^B_x(f \circ k)$ for any monotone $k : I \to A$.

Proposition 3.7 *$f : A \to B$, a monotone map between dcpos, is externally a dcpo map iff it is internally a dcpo homomorphism.*

Proof. If it is internally a dcpo homomorphism then $f \circ [\Sigma_x^A(k)] = \Sigma_x^B(f \circ k)$ since we have constructed Σ_x explicitly in terms of the directed join map $\bigvee^{\uparrow} : idl(A) \to A$.

Conversely if f is an external dcpo map then $f \circ \bigvee^{\uparrow}_A = f \circ \Sigma^A_{n_A}(e_A)$ where $\in_A \stackrel{(n_A,e_A)}{\to} idlA \times A$. Therefore $f \circ \bigvee^{\uparrow}_A = \Sigma^B_{n_A}(f \circ e_A)$. But the explicit formula given for $\Sigma^B_{n_A}$ shows that for any ideal I of A,

$$\begin{aligned}\Sigma^B_{n_A}(f \circ e_A)(I) &= \bigvee\nolimits^{\uparrow}_B \{f \circ e_A(i) \mid n_A(i) = I\} \\ &= \bigvee\nolimits^{\uparrow}_B \{f(a) \mid a \in I\}.\end{aligned}$$

□

4 The direct image of dcpos

So far we have only considered situations with respect to a single topos $\mathcal{E}$. The aim of this section is to introduce the action of geometric morphisms on the structures in question. In particular to show that, for any geometric morphism $f : \mathcal{E} \to \mathcal{E}'$,

$$f_* : \mathbf{dcpo}_{\mathcal{E}} \to \mathbf{dcpo}_{\mathcal{E}'}$$

is well defined where f_* is the direct image part of f and of course the notation $\mathbf{dcpo}_{\mathcal{E}}$ is for the category of dcpos internal to the topos $\mathcal{E}$.

Now, given A, a semilattice in a topos $\mathcal{E}$ and a geometric morphism $f : \mathcal{E} \to \mathcal{E}'$, it is known that f_*A is a semilattice internal to $\mathcal{E}'$. This is immediate since f_* preserves products and the property of being an internal semilattice is expressible using diagrams involving only products. In contrast it is a little harder to show that the property of simply being a poset is also preserved by the direct image part of any geometric morphism.

Lemma 4.1 *If $(G, \leq)$ is a poset in a topos $\mathcal{E}$ and $f : \mathcal{E} \to \mathcal{E}'$ a geometric morphism then $(f_*G, f_* \leq)$ is a poset in $\mathcal{E}'$. Further $f_*(G^{op}) \cong (f_*G)^{op}$.*

Proof. The property of being an internal category is certainly preserved by f_*, since, for example, if $\circ :\leq \times_G \leq \to \leq$ is composition then $f_*\circ : f_* \leq \times_{f_*G} f_* \leq \to f_* \leq$ is composition (internal to $\mathcal{E}'$) as f_*commutes with pullbacks. The property of being monic is preserved as all monics are regular in a topos and so f_*G is an internal preorder from the definitions of the previous section. Finally a poset is a preorder such that G is the pullback of $\leq$ along $\geq$; and so the result follows by the preservation of pullbacks.

That $f_*(G^{op}) \cong (f_*G)^{op}$ is immediate from the definitions. □

Exactly the same argument as used in this lemma applies to the inverse image of a geometric morphism (since it too, by definition, preserves finite limits). If $(G, \leq)$ is a poset in a topos $\mathcal{E}'$ then $(f^*G, f^* \leq)$ is a poset in $\mathcal{E}$ and so we have almost proved:

Lemma 4.2 *The property of being*
(i) a poset
(ii) a monotone map and
(iii) a fiber directed monotone map
are each preserved by the inverse image of any geometric morphism.

Proof. (i) Done.

(ii) Immediate as f^* preserves binary products (and commuting squares!).

(iii) Note that fiber directedness for $x : I \to J$ is the assertion that (a) x and $\pi_{13} : \leq_I \times_J \geq_I \to I \times_J I$ are both regular epimorphisms (i.e. surjections) and (b) there exists a map

$$\delta : [I \times_J J] \times_J \leq_J \to \leq_I \times_I [I \times_J J])$$

$\pi_1 \circ \delta = \pi_1$ and $\pi_4 \circ \delta = \pi_4$. The property of being a regular epimorphism is preserved by the inverse image of any geometric morphism, and so the result follows since the pullbacks involved in the definition are also preserved. $\square$

The next lemma is trivial but pivotal since it shows how monotone maps translate from one topos to another. Most of the rest of the paper concerns itself with specializing this lemma.

Lemma 4.3 *For any poset I' in $\mathcal{E}'$ and any poset A in $\mathcal{E}$ given a geometric morphism $f : \mathcal{E} \to \mathcal{E}'$ there is an order isomorphism*

$$\mathbf{Pos}_{\mathcal{E}'}(I', f_*A) \cong \mathbf{Pos}_{\mathcal{E}}(f^*I', A)$$

(natural in I' and A) specializing the bijection $\mathcal{E}'(I', f_*A) \cong \mathcal{E}(f^*I', A)$.

Proof. $i' : I' \to f_*A$ is monotone iff there exists $n : \leq_{I'} \to f_*(\leq_A)$ such that

$$\begin{array}{ccc} \leq_{I'} & \xrightarrow{n} & f_* \leq_A \\ \downarrow & & \downarrow \\ I' \times I' & \xrightarrow{i' \times i'} & f_*A \times f_*A \end{array}$$

commutes. By taking the adjoint transpose of this square it is clear that i' is monotone iff i is monotone where i is the adjoint transpose of i'. Similarly $i'_1 \leq i'_2$ in the external poset $\mathbf{Pos}_{\mathcal{E}'}(I', f_*A)$ iff $i_1 \leq i_2$ in $\mathbf{Pos}_{\mathcal{E}}(f^*I', A)$. Naturality is immediate from the naturality of $\mathcal{E}'(\_, f_*\_) \cong \mathcal{E}(f^*\_, \_)$. $\square$

This will be used in the proof of the next proposition.

Proposition 4.4 *If A is a dcpo in a topos $\mathcal{E}$ and $f : \mathcal{E} \to \mathcal{E}'$ a geometric morphism then f_*A is a dcpo in $\mathcal{E}'$.*

Proof. To prove that f_*A is a dcpo in $\mathcal{E}'$ is must be verified that for every fiber directed map $x' : I' \to J'$, there exists $\Sigma_{x'}$ left adjoint to x'^* such that the Beck-Chevalley conditions hold over all such adjoint pairs. But, if x' is fiber directed, then so is $f^*x' : f^*I' \to f^*J'$ in $\mathcal{E}$. Naturality of $\mathbf{Pos}_{\mathcal{E}'}(I', f_*A) \cong \mathbf{Pos}_{\mathcal{E}}(f^*I', A)$ with respect to x' shows that $(x')^*$ factors through $(f^*x')^*$ via two isomorphisms and so $(x')^*$ has a left adjoint if $(f^*x')^*$ does.

The inverse image of a pullback of fiber directed maps is a pullback of fiber directed maps and so the Beck-Chevalley conditions (for f_*A) follow as the functions $\Sigma_{x'}, \Sigma_{f^*x'}, (x')^*$ and $(f^*x')^*$ commute with the isomorphisms $\mathbf{Pos}_{\mathcal{E}'}(\_, f_*A) \cong \mathbf{Pos}_{\mathcal{E}}(f^*\_, A)$. □

Theorem 4.5 *Given a geometric morphism $f : \mathcal{E} \to \mathcal{E}'$ its direct image part defines a functor $f_* : \mathbf{dcpo}_{\mathcal{E}} \to \mathbf{dcpo}_{\mathcal{E}'}$.*

Proof. This has just been defined on objects. It extends to morphism by naturality of $\mathbf{Pos}_{\mathcal{E}'}(I', f_*A) \cong \mathbf{Pos}_{\mathcal{E}}(f^*I', A)$ in the second component. □

5 The inverse image of dcpo presentations

As indicated in the introduction, the left adjoint to $f_* : \mathbf{dcpo}_{\mathcal{E}} \to \mathbf{dcpo}_{\mathcal{E}'}$ will be found by applying the inverse image of f to dcpo presentations. The fact that this is well defined is relatively easy to verify.

Lemma 5.1 *If $f : \mathcal{E} \to \mathcal{E}'$ is a geometric morphism and (G', R', λ', π') a dcpo presentation in $\mathcal{E}'$ then*

$$(f^*G', f^*R', f^*\lambda', f^*\pi')$$

is a dcpo presentation in $\mathcal{E}$.

Proof. The only difficulty is showing that for every $r \in f^*R$ the set (object of $\mathcal{E}$) given by $\{g \in f^*G \mid gf^*\pi' r\}$ is directed lower closed, but this is just the assertion that $f^*\pi'$ is an approximable mapping (with f^*R a discrete poset since R is). But, just as in the proof that fiber directed maps are stable under f^*, it can be shown that approximable mappings are stable under f^*. For example the assertion that $\{a \in A \mid aRb\}$ is directed for every b is equivalent to insisting that the two maps $\pi_2 : R \to B$ and $forget : \leq_A \times_A \leq_A \times_A R \to R \times_B R$ are both regular epimorphisms (surjections), where $forget(a', a''', a'', a''', a''', b) = (a', b, a'', b)$ and $\leq_A \times_A \leq_A \times_A R = \{(a', a''', a'', a''', a''', b) \mid a' \leq a''', a'' \leq a''', a''' Rb\}$. The constructions involved (pullback, saying that certain diagrams commute, that certain maps exist and that certain maps are regular epimorphisms) are all stable under f^*. □

Logicians may not like the above proof as it uses categorical language to obscure a proof which essentially follows from the observation that dcpo presentations are models of geometric theories.

6 The left adjoint to $f_* : \mathbf{dcpo}_{\mathcal{E}} \to \mathbf{dcpo}_{\mathcal{E}'}$

We are now in a position to state the key theorem leading to the main result. This theorem gives a universal description of the dcpo that is being presented by the inverse image of any dcpo presentation.

Theorem 6.1 *If $(G', R', ...)$ is a dcpo presentation in $\mathcal{E}'$ then, for any $\mathcal{E}$ dcpo A there is a bijection between monotone maps $\phi' : G' \to f_*A$ satisfying the relations R' and monotone maps $\phi : f^*G' \to A$ satisfying the relations f^*R'. Further this correspondence is natural in dcpo maps between As and monotone maps between G's.*

Proof. The adjunction of the geometric morphism f, by definition, sets up a bijection. What remains to check is that under this bijection maps satisfying R' correspond to maps satisfying f^*R'. (That the property of being a monotone map is preserved under the bijection has been covered already.)

When we say that $\phi' : G' \to f_*A$ "satisfies R'" we are stating that $\phi' \circ \lambda' = \bigvee^{\uparrow}\{\phi'(l') \mid l'\pi'(\_)\}$ in the homset $R' \to f_*(A)$. The adjoint transpose of $\phi' \circ \lambda'$ is $\phi \circ f^*(\lambda')$ and so the proof will be completed provided that we can argue that the adjoint transpose of $\bigvee^{\uparrow}\{\phi'(l') \mid l'\pi'(\_)\} : R' \to f_*(A)$ is $\bigvee^{\uparrow}\{\phi(l) \mid l(f^*\pi')(\_)\} : f^*R' \to A$. However from the explicit construction of Σ_x for any (fiber directed) re-indexing map $x : I \to J$ given in the proof of Lemma 3.5, it is clear that $\Sigma_{\pi_2}(\pi' \xrightarrow{\pi_1} G' \xrightarrow{\phi'} f_*(A)) = \bigvee^{\uparrow}\{\phi'(g') \mid g'\pi'(\_)\}$ for $\pi_2 : \pi' \to R'$ and similarly $\Sigma_{f^*\pi_2}(f^*\pi' \xrightarrow{\pi_1} f^*G \xrightarrow{\phi} A) = \bigvee^{\uparrow}\{\phi(g) \mid gf^*\pi'(\_)\}$.

The proof of Proposition 4.4 shows that Σ_{π_2} is the same map as $\Sigma_{f^*\pi_2}$ modulo the poset isomorphism given by the adjunction and so the result follows. Naturality is immediate from the naturality of the adjunction that defines the geometric morphism. $\Box$

From this observation the main result for this paper is immediate.

Theorem 6.2 $f_* : \mathbf{dcpo}_{\mathcal{E}} \to \mathbf{dcpo}_{\mathcal{E}'}$ *has a left adjoint.*

Proof. Every dcpo has a canonical presentation (Example 2.2) and so the bijection just established shows that for every dcpo A' in $\mathcal{E}'$, there exists $f^{\#}(A')$ a dcpo in $\mathcal{E}$ and a natural isomorphism

$$\mathbf{dcpo}_{\mathcal{E}}(f^{\#}(A'), \_) \cong \mathbf{dcpo}_{\mathcal{E}'}(A', f_*(\_)).$$

If $g' : A' \to B'$ is a dcpo map in $\mathcal{E}'$ then $f^{\#}(g') : f^{\#}A' \to f^{\#}B'$ is the mate of $g' : A' \to B' \to f_*f^{\#}(B')$ under this isomorphism where the second map (the unit) is the mate of $Id : f^{\#}B' \to f^{\#}B'$. It is then routine to verify that $f^{\#}$ is (a functor and) left adjoint to f_* as required. $\Box$

It is worth making explicit the 2-categorical nature of the adjunction $f^{\#} \dashv f_*$.

Lemma 6.3 *(a) The bijection* $\mathbf{dcpo}_{\mathcal{E}}(f^{\#}A', B) \cong \mathbf{dcpo}_{\mathcal{E}'}(A', f_*B)$ *is an order isomorphism.*

(b) On morphisms $f^{\#} : \mathbf{dcpo}_{\mathcal{E}'}(A', B') \rightarrow \mathbf{dcpo}_{\mathcal{E}}(f^{\#}A', f^{\#}B')$ *preserves external order.*

Proof. (a) Immediate from the initial observation (Lemma 4.3) that $\mathbf{Pos}_{\mathcal{E}}(f^*A', B) \cong \mathbf{Pos}_{\mathcal{E}'}(A', f_*B)$ is an order isomorphism. (The universal bijection $\{\phi \in \mathbf{Pos}_{\mathcal{E}}(G, B) \mid \phi \text{ satisfies } R\} \cong \mathbf{dcpo}_{\mathcal{E}}(dcpo\langle G \mid R\rangle, B)$ preserves order.)

(b) $f^{\#}(q')$ is the adjoint transpose of $\eta_{B'} \circ q'$ (for $q' : A' \rightarrow B'$) and so this follows from (a) since function composition preserves order in each component. □

7 Frames as distributive lattices over dcpos

The next objective will be to extend the main result to locales and this will be done by exploiting the fact that the adjunction $f^{\#} \dashv f_*$ is order enriched. For background on locales consult [Johnstone 82]; the category of locales (**Loc**) is the opposite of the category of frames (**Fr**). A frame is a complete lattice such that finite meets distribute over arbitrary joins. Frame homomorphisms preserves arbitrary joins and finite meets.

Now some new results are developed which show how frames can be viewed as distributive lattices over dcpos.

Definition 7.1 If **C** is an order enriched category with finite products then define $DLat(\mathbf{C})$ as the *order-internal distributive lattices on* **C**. Its objects are 4-tuples $(L, \vee : L \times L \rightarrow L, \wedge : L \times L \rightarrow L, 0_L : 1 \rightarrow L, 1_L : 1 \rightarrow L)$ such that:

(i) $\vee$ is left adjoint to the diagonal (in the order enrichment), and 0_L is left adjoint to $! : L \rightarrow 1$,

(ii) $\wedge$ is right adjoint to the diagonal, and 1_L is right adjoint to $! : L \rightarrow 1$ and

(iii) $\wedge$ distributes over $\vee$ in the usual manner.

The morphisms of $DLat(\mathbf{C})$ are those morphisms of **C** which commute with the operations $\vee, \wedge, 0_L$ and 1_L in the usual manner.

Lemma 7.2 **Fr** $\cong DLat(\mathbf{dcpo})$.

Proof. Firstly it is easy to check that the definition of $\vee$ and $\wedge$ is sufficient to prove that they are indeed the join and meet operation for the underlying poset of any dcpo. Then this central lemma is actually immediate since it has been established already that dcpo product is tensor. It follows that the binary meet map distributes over directed joins (that the join map distributes over finite meets is immediate from the axiom of being a distributive lattice). □

Now $f_* : \mathbf{dcpo}_{\mathcal{E}} \rightarrow \mathbf{dcpo}_{\mathcal{E}'}$ preserves the external ordering on homsets and finite products. So it certainly preserves the property of being an order-internal distributive lattice. Hence there is a restriction $f_* : DLat(\mathbf{dcpo}_{\mathcal{E}}) \rightarrow$

$DLat(\mathbf{dcpo}_{\mathcal{E}'})$. It has also been observed that $f^{\#}:\mathbf{dcpo}_{\mathcal{E}'}\rightarrow\mathbf{dcpo}_{\mathcal{E}}$ preserves the homset ordering. Therefore it has almost been shown that:

Proposition 7.3 *$f_*:DLat(\mathbf{dcpo}_{\mathcal{E}})\rightarrow DLat(\mathbf{dcpo}_{\mathcal{E}'})$, i.e. $f_*:\mathbf{Fr}_{\mathcal{E}}\rightarrow\mathbf{Fr}_{\mathcal{E}'}$ has a left adjoint, given by $f^{\#}:\mathbf{dcpo}_{\mathcal{E}'}\rightarrow\mathbf{dcpo}_{\mathcal{E}}$.*

Proof. It must only be verified that $f^{\#}$ preserves finite products. It certainly preserves 1 since the final dcpo is the singleton set $\{*\}$ and this is presented by itself. f^* preserves the singleton set. As for binary products it has been shown that these are tensor. But tensor has been defined via its presentation and so it must be verified that the image of an arbitrary tensor presentation under f^* is again the presentation of tensor product. But by using Proposition 2.5 it can be seen that the explicit presentation given for dcpo tensor is stable under $f^*:\mathcal{E}'\rightarrow\mathcal{E}$ since f^* preserves coproduct and product. For example, using the notation of that proposition, note that $\pi_{\otimes}=\pi_L\amalg\pi_R$, $\pi_L\cong\leq_{G_1}\times\pi_2$ and $\pi_R\cong\leq_{G_2}\times\pi_1$and so these are preserved by f^*. □

The main insights are now complete. The category of dcpos is good enough to carry the data of frames from one topos to another. The remainder of the paper looks at how this works out in practice.

7.1 Presenting frames

We will define the notion of a *DL-site* which is a type of presentation for a frame. In a DL-site the generators form a distributive lattice (DL) and the relations, involving only directed joins, must have both meet and join stability. To express the meet and join stability properties succinctly we use the idea of an L-set for any distributive lattice L. This is simply a set with two actions by L, for the monoids $(L,0,\vee)$ and $(L,1,\wedge)$.

Example 7.4 The set $idl(L)$ is an L-set with actions

$$\begin{aligned}(l,I)&\longmapsto\{l\wedge m\mid m\in I\}\\(l,I)&\longmapsto\downarrow\{l\vee m\mid m\in I\}\end{aligned}$$

Definition 7.5 A *DL-site* comprises a distributive lattice L, an L-set R and a pair of L-set homomorphisms $e_1,e_2:R\rightrightarrows idl(L)$ such that (a) e_2 factors through $\downarrow:L\rightarrow idl(L)$ and (b) $e_1\leq e_2$.

Meet and join stability is the assertion that the maps e_1,e_2 are L-set homomorphisms, see [JoVic 91]. For example meet stability is the statement that if

$$\lambda(r)=\bigvee^{\uparrow}\{l'\in L\mid l'\pi r\}$$

is universally true in the frame being presented for every r in R, then for any $l\in L$,

$$\lambda(r) \wedge l = \bigvee^{\uparrow}\{l' \wedge l \mid l' \in L, l'\pi r\},$$

will also be in R, where, we are writing $l'\pi r$ iff $l' \in e_1(r)$ and are assuming that $e_2 = \downarrow \circ \lambda$ for $\lambda : R \to L$. DL-sites, of course, also present dcpos by forgetting the distributive lattice structure. In other words:

Lemma 7.6 *Every DL-site is a dcpo presentation.*

Proof. e_2 factors via some $\lambda : R \to L$ and $\pi \subseteq L \times R$ is given by $l'\pi r$ iff $l' \in e_1(r)$. □

It is just a convenience that here we are insisting that e_2 factors via $\downarrow$ and that $e_1 \leq e_2$; it can be shown that any presentation without this assumption presents the same frame as one with this assumption. Moreover, the canonical example of a DL-site always satisfies this assumption.

Example 7.7 Any frame has a presentation by a DL-site. Given a frame ΩX, take $L^X = \Omega X$ and $R^X = idl(\Omega X)$. The L^X-set morphisms from R^X to $idl(L^X)$ are the identity and $\downarrow \circ \bigvee^{\uparrow}$. Such a presentation is referred to as the *standard presentation* for the frame

(The notation ΩX is standard for the frame corresponding to the locale X, consult, e.g. [Johnstone 82].) Although it will be useful to give an explicit description (in terms of C-ideals) of frames from their presentations (this is for the main application of the paper) it is of some interest to note that we do not initially need to know that such a description can be given. That DL–sites present frames (i.e. that free frames exist on the generators qua relations) can be deduced from the fact that dcpo presentations present, since:

Theorem 7.8 (double coverage theorem) *If* $(L, R, ...)$ *is a DL-site, then*

$$\mathbf{Fr}\langle L \text{ (qua DL)} \mid R\rangle \cong \mathbf{dcpo}\langle L \text{ (qua poset)} \mid R\rangle$$

Proof. The right-hand side is defined; let us denote it by A. $A \times A \cong A \otimes_{\mathbf{dcpo}} A$ (from above) and $A \otimes_{\mathbf{dcpo}} A$ is generated by $L \times L$. But, by the join stability assumption, $L \times L \xrightarrow{\vee} L \xrightarrow{i_A} A$ satisfies the relations involved in presenting $A \otimes_{\mathbf{dcpo}} A$ (where i_A is the universal map) and so a map $A \times A \to A$ is defined, which can be verified to be join. Similarly A has finite meets, and inherits finite distributivity from L. It is then easy to show that A does indeed have the universal property required by the left-hand side. □

The double coverage theorem also appears in [TowVic 02].

In contrast to dcpos presentations, we can give an explicit description of the opens of any frame given a DL-site presenting it. This was first made very clear by Johnstone in his coverage theorem. Importantly the explicit description given here in terms of DL-sites does not break the symmetry between finite joins and finite meets, so whilst this basic result is not new, it does offer new insight into how to maintain the preframe/suplattice symmetry when discussing locales (see e.g. [Townsend 96] for a discussion of this symmetry)

Theorem 7.9 *If L is a distributive lattice and $R \rightrightarrows L$ is a meet and join stable collection of directed relations on L (i.e. the data for a DL-site) then*

$$\Omega X \equiv \mathbf{Fr}\langle L \textit{ qua DLat} \mid R\rangle$$

is isomorphic to the set of ideals I of L with the property that if $\{l \mid l\pi r\} \subseteq I$ then $\lambda(r) \in I$.

The sets of ideals which satisfy this property (i.e. $\{l \mid l\pi r\} \subseteq^{\uparrow} I \implies \lambda(r) \in I$)) are, following Johnstone, called *C-ideals*, and the set of all such C-ideals (given a DL-site $(L, R, ..)$) is denoted $C - Idl(L)$. However, please note that the notation "C" is not used as part of the definition of the notion of coverage used here.

Proof. L is a meet-semilattice and the relations (including the "qua join semilattice" ones) are meet stable. Therefore Johnstone's original coverage theorem applies; Section 2.11, Ch. II of [Johnstone 82]. It is immediate that $C-Idl(L)$ is a complete lattice since it is closed under arbitrary intersections. It is also a Heyting algebra (you may check using basic lattice theory that a frame is exactly a complete Heyting algebra); if I, J are C-ideals then

$$I \rightarrow J = \{k \in L \mid k \wedge i \in J \quad \forall i \in I\}$$

is a C-ideal, this is from the meet stability of the DL-site. That the C-ideals satisfy the correct universal properties is a straightforward verification and widely known. □

Remark 7.10 In the standard presentation for a frame, all C-ideals are principal. To see this, say $I \subseteq \Omega X$ is a C-ideal. Then I is an ideal and so in the standard presentation I covers $\bigvee^{\uparrow} I$. Hence $\bigvee^{\uparrow} I \in I$ by definition of C-ideal (since, certainly, $I \subseteq I$) and so I is the principal ideal $\downarrow \bigvee^{\uparrow} I$.

The next proposition shows that the explicit description of opens given by C-ideals can actually be re-expressed in terms of satisfying relations, and so the techniques developed to translate this property between toposes can be applied.

Proposition 7.11 *Given ΩX presented by the DL-site $(L^X, R^X, ...)$, the opens of ΩX are exactly the join semilattice homomorphisms $L^X \rightarrow \Omega^{op}$ which satisfy R^X where Ω is the subobject classifier.*

Proof. $\chi_I : L^X \rightarrow \Omega^{op}$ satisfies R means, for each $r \in R^X$,

$$\chi_I \lambda(r) = \bigwedge\nolimits_{\Omega} \{\chi_I(l) \mid l \in L^X, \, l\pi r\}.$$

I.e. if $\{l \mid l\pi r\} \subseteq I$ then $\lambda(r) \in I$.

An ideal on L^X is exactly a join semilattice homomorphism $L^X \to \Omega^{op}$. □

The above descriptions have not used the join stability of the DL-site, but the next subsection will exploit it and thereby provide a shortcut to the applications of this paper, the existence of which is of some technical interest.

7.2 *Shortcut to the applications*

In all the applications offered here we do not investigate dcpos in their own right. We are only interested in viewing the dcpo homomorphisms (i.e. Scott continuous maps) away from frames. The logic adopted above has been: (i) show that dcpo presentations present and then (ii) show that, via the double coverage result, the dcpos presented by DL-sites are in fact frames (and further all frames may be described in this way). By this method dcpo morphisms away from frames can be described. However given the explicit description of the frame presented by a DL-site in terms of C-ideals (last section, and widely known) there is a shortcut to the double coverage result which can be used to prove all the applications below.

Theorem 7.12 *If $(L, R, ...)$ is a DL-site, then*

$$\mathbf{Fr}\langle L \textit{ (qua DL)} \mid R\rangle \cong \mathbf{dcpo}\langle L \textit{ (qua poset)} \mid R\rangle.$$

Proof. It has been shown that $\mathbf{Fr}\langle L$ (qua DL)$\mid R\rangle \cong C - Idl(L)$; this uses the meet stability of the DL-site presentation of the frame. Given a subset $J \subseteq L$ define $C - Idl < J >= \cap\{\overline{J} \mid \overline{J}$ a C-ideal, $J \subseteq \overline{J}\}$, i.e. the C-ideal closure of J. Recall that a C-ideal, I, is an ideal (i.e. lower closed directed) which satisfies $\{l \mid l\pi r\} \subseteq I$ implies $\lambda(r) \in I$. The map $L \to C - Idl(L)$ given by $l \longmapsto C - Idl <\downarrow l >$ (is monotone and) satisfies the relations R. To see this note that if $(J_i)_{i \in I}$ is some indexed collection of C-ideals then

$$\bigvee_{C-Idl(L)} \{J_i \mid i \in I\} = C - Idl < \bigcup_{i \in I} J_i > .$$

Now, if the indexing set is directed then there another description of this join:

$$\bigvee^{\uparrow}_{C-Idl(L)} \{J_i \mid i \in I\} = wC - Idl < \bigcup^{\uparrow}_{i \in I} J_i > ,$$

where $wC - Idl(L)$ is the set of weak C-ideals. A weak C-ideal is a lower closed subset such that $\{l \mid l\pi r\} \subseteq I$ implies $\lambda(r) \in I$ for every relation r. In other words a weak C-ideal is a C-ideal which is not necessarily an ordinary ideal. To prove this new description of directed join in $C - Idl(L)$ it is sufficient to show that $wC - Idl < K >\equiv \cap\{\overline{J} \mid \overline{J}$ a weak C-ideal, $K \subseteq \overline{J}\}$ is an ideal if K is (since, $\bigcup^{\uparrow}_{i \in I} J_i$, you may verify, is an ideal). Certainly $wC - Idl < K >$ is lower closed, and it is non-empty since K is. It remains to prove that given some $k_1, k_2 \in wC - Idl < K >$ that $k_1 \vee k_2 \in wC - Idl < K >$. Consider the set $K_1 \equiv \{k \in wC - Idl < K >| \ \forall l \in K, \ k \vee l \in wC - Idl < K >\}$. Now, by the join stability assumption on the DL-site, K_1 is a weak C-ideal. Now, certainly $K \subseteq K_1$ since K is an ideal, and so by taking weak C-ideal closure we see that $wC - Idl < K > \subseteq K_1$. Next, set $K_2 \equiv \{k \in wC - Idl < K >| \ \forall l \in wC - Idl < K >, \ k \vee l \in wC - Idl < K >\}$. This also is a weak C-ideal by join stability. Also, $K \subseteq K_2$ by the fact that $wC - Idl < K > \subseteq K_1$, and so by taking weak C-ideal closure it is demonstrated that $wC - Idl < K > \subseteq K_2$, i.e. $wC - Idl < K >$ is closed under binary join and so is an ideal.

The hard part of the proof is completed and the rest is straightforward verification which is included for completeness. Firstly, clearly,

$$J = \bigvee_{C-Idl(L)} \{C - Idl <\downarrow l >| \ l \in J\}$$

for any C-ideal J and since J is a directed set this join is directed. It therefore follows that if $\phi : L \to B$ is a monotone map to a dcpo B which satisfies the relations R then, if $\phi = q \circ C - Idl <\downarrow \_ >$, for some dcpo map $q : C - Idl(L) \to B$, q must be given by

$$q(J) = \bigvee\nolimits^{\uparrow}_{B} \{\phi(l) \mid l \in J\}.$$

To complete the proof it remains to check that this assignment (i.e. using this to define a function q) is a dcpo homomorphism. The hard bit of proving this fact is showing that

$$q(\bigvee\nolimits^{\uparrow}_{C-Idl(L)} \{J_i \mid i \in I\}) \leq \bigvee\nolimits^{\uparrow}_{B} \{q(J_i) \mid i \in I\},$$

i.e.,

$$\bigvee\nolimits^{\uparrow}_{B} \{\phi(l) \mid l \in wC - Idl < \bigcup\nolimits^{\uparrow}_{i \in I} J_i >\} \leq \bigvee\nolimits^{\uparrow}_{B} \{\phi(l') \mid l' \in \bigcup\nolimits^{\uparrow}_{i \in I} J_i\}$$

for any directed collection $(J_i)_{i \in I}$ of C-ideals, where the weak ideal closure is appropriate by the result just shown on directed joins in $C - Idl(L)$. The result will then follow if it can be shown that $K \equiv \{l \in L \mid \phi(l) \leq \bigvee^{\uparrow}_{B} \{q(J_i) \mid i \in I\}\}$ is a weak C-ideal, since certainly $\bigcup^{\uparrow}_{i \in I} J_i \subseteq \{l \in L \mid \phi(l) \leq \bigvee^{\uparrow}_{B} \{q(J_i) \mid i \in I\}\}$. But given that ϕ satisfies R, then for any $r \in R$ it is clear that if $\{l \mid l \pi r\} \subseteq K$ then $\lambda(r) \in K$ and so this is immediate. □

Thus the reader may not wish to concern himself with the question of whether dcpo presentations present since the above result shows that DL-sites always present dcpos and in our applications we only wish to present dcpos via DL-sites. This observation may be applicable to the generalization of this work to toposes; see the concluding comments below.

Of course whether or not this route is better is open to debate, given the detail needed in the proof above. One also needs a little extra work to ensure that $f_* : \mathbf{Fr}_\mathcal{E} \to \mathbf{Fr}_{\mathcal{E}'}$ has a left adjoint without the knowledge that dcpo tensor presentations present. This extra work is just a question of verifying that the inverse image of DL-sites are DL-sites (and that the equivalence of Theorem 6.1 restricts to distributive lattice homomorphisms). Both these observations are straightforward given the techniques developed.

8 The Joyal and Tierney correspondence

Before we look at applications it is worth recalling the well known Joyal and Tierney correspondence between locales internal in a topos of sheaves of a locale Y and the slice of locales over Y. To prove that the left adjoint to $f_* : \mathbf{Fr}_\mathcal{E} \to \mathbf{Fr}_{\mathcal{E}'}$ is indeed locale pullback this correspondence will be needed.

Theorem 8.1 *For any locale Y in a topos $\mathcal{E}$ there is a geometric morphism $! : Sh(Y) \to \mathcal{E}$, from the topos of sheaves over Y to $\mathcal{E}$ with the property that $!_*\Omega_{Sh(Y)} \cong \Omega_\mathcal{E} Y$. Further the map $\mathbf{Fr}_{Sh(Y)} \to !_*\Omega_{Sh(Y)}/\mathbf{Fr}_\mathcal{E}$ (i.e. to the coslice), given by sending any $\Omega_{Sh(Y)}X$ to $!_*\Omega!^X$ is part of an equivalence and so $\mathbf{Loc}_{Sh(Y)} \cong \mathbf{Loc}_\mathcal{E}/Y$.*

Proof. [JoyTie 84]. In any topos Ω is the initial frame and so the map is well defined. ($\Omega!^X : \Omega \to \Omega X$ is standard notation for the unique map from the initial frame, as it dualises the locale map $!^X : X \to 1$.) □

Notation warning: ! is used both as a geometric morphism and as a locale map.

Theorem 8.2 *(The left adjoint $f^\#$ is locale pullback) If $f : X \to Y$ is a locale map then for any $p : Z \to Y$ the pullback of p along f is given by $f^\# \Omega_{Sh(Y)} Z_p$ where Z_p is the locale corresponding to $p : Z \to Y$ in $Sh(Y)$ and f is identified with the geometric morphism $f : Sh(X) \to Sh(Y)$.*

Proof. From the proof of the Joyal-Tierney correspondence it is evident that $f_* : \mathbf{Fr}_{Sh(X)} \to \mathbf{Fr}_{Sh(Y)}$ is "precompose with Ωf". But f_* has a left adjoint and so the action of f_* on locales (which is $\Sigma_f : \mathbf{Loc}/X \to \mathbf{Loc}/Y$, "post compose with f") has a right adjoint, i.e. pullback. □

It will also ease proofs to follow to have an explicit description of the of the inverse image f^*I of a subset I in terms of its classifying map. This is a completely general result about the inverse image of any monomorphism and so is of some independent interest.

Lemma 8.3 *If $f : \mathcal{E} \to \mathcal{E}'$ is any geometric morphism and $i : A_0 \hookrightarrow A$ is a monomorphism in $\mathcal{E}'$ classified by $\chi_i : A \to \Omega_{\mathcal{E}'}$ then the monomorphism $f^*i : f^*A_0 \hookrightarrow f^*A$ in $\mathcal{E}$ is classified by $v \circ f^*\chi_i$ where $v : f^*\Omega_{\mathcal{E}'} \to \Omega_\mathcal{E}$ is the adjoint transpose of the unique frame homomorphism $\Omega_{\mathcal{E}'}! : \Omega_{\mathcal{E}'} \to f_*\Omega_\mathcal{E}$.*

Proof. By the uniqueness of classifying maps (in the definition of subobject classifier) this amounts to showing that the outer rectangle in

$$\begin{array}{ccccc} f^*A_0 & \longrightarrow & f^*1 & \stackrel{\cong}{\longrightarrow} & 1 \\ \downarrow f^*i & & f^*\top \downarrow & & \downarrow \top \\ f^*A & \stackrel{f^*(\chi_i)}{\rightarrow} & f^*\Omega_{\mathcal{E}'} & \stackrel{v}{\rightarrow} & \Omega_{\mathcal{E}} \end{array}$$

is a pullback, where v is the adjoint transpose of $\Omega_{\mathcal{E}'}! : \Omega_{\mathcal{E}'} \hookrightarrow f_*\Omega_{\mathcal{E}}$. Since f^* preserves pullbacks, the left square is a pullback and it remains to show that the right hand square is also a pullback. Now, certainly there does exist some $w : f^*\Omega_{\mathcal{E}'} \to \Omega_{\mathcal{E}}$ which pulls $\top$ back to $f^*\top$ (this is because $f^*\top : f^*1 \hookrightarrow f^*\Omega_{\mathcal{E}'}$ is monic and so has a classifying map), we must show that the adjoint transpose of w is $\Omega_{\mathcal{E}'}!$. In fact by taking the adjoint transpose of the square $w \circ f^*\top = \top \circ \cong$ it can be seen that $w'(\top) = \top_{f_*\Omega_{\mathcal{E}}}$, where w' is the adjoint transpose of w. Since $\Omega_{\mathcal{E}}$ is (well known to be) the free suplattice on the singleton set 1 and since $\Omega_{\mathcal{E}'}!$ could have equivalently been defined as the unique suplattice homomorphism that sends the element of the singleton 1 to $\top_{f_*\Omega_{\mathcal{E}}}$ all that remains is to be sure the w' is a suplattice homomorphism. This indeed it is, since we can define $z' : f_*\Omega_{\mathcal{E}} \to \Omega_{\mathcal{E}'}$ right adjoint to w'. Set z' to be the classifying map of $f_*\top_{\Omega_{\mathcal{E}}}$. Then $\top_{\Omega_{\mathcal{E}'}}$ is contained in the subobject of $\Omega_{\mathcal{E}'}$ classified by $z' \circ w'$ since w' can be factored as $f_*w \circ \eta_{\Omega_{\mathcal{E}'}}$ and the pullback of $f_*\top_{\Omega_{\mathcal{E}}}$ along f_*w is $f_*f^*\top_{\Omega_{\mathcal{E}'}}$ as f_* preserves pullback. Hence $Id \leq z' \circ w'$ in $\mathbf{Pos}_{\mathcal{E}'}(\Omega_{\mathcal{E}'}, \Omega_{\mathcal{E}'})$. The adjoint transpose of $w' \circ z'$ classifies $f^*f_*\top_{\Omega_{\mathcal{E}}}$ since f^* preserves pullbacks (i.e. the classifying pullback that defined z') and so since the subobject $f^*f_*\top_{\Omega_{\mathcal{E}}} : f^*f_*1 \hookrightarrow f^*f_*\Omega_{\mathcal{E}}$ is contained in the subobject classified by $\epsilon_{\Omega_{\mathcal{E}}} : f^*f_*\Omega_{\mathcal{E}} \to \Omega_{\mathcal{E}}$ is follows that the adjoint transpose of $w' \circ z'$ is less than $\epsilon_{\Omega_{\mathcal{E}}}$ in the poset $\mathbf{Pos}_{\mathcal{E}}(f^*f_*\Omega_{\mathcal{E}}, \Omega_{\mathcal{E}})$ and so $w' \circ z' \leq Id$.

That this external statement of having a right adjoint is enough to show that w' is a suplattice homomorphism internally (i.e. that $\bigvee_{f_*\Omega_{\mathcal{E}}} \circ P(w') = w' \circ \bigvee_{\Omega_{\mathcal{E}}}$) is well known and can be easily derived from part (i) of Lemma 3.5. □

(Acknowledgment is due to Prof. Johnstone for pointing out the simple version of this proof.) Similarly, here is another easy topos theoretic result needed.

Lemma 8.4 *If $f : X \to Y$ is a locale map (in some topos $\mathcal{E}$) then (also using the notation $f : Sh(X) \to Sh(Y)$ for the corresponding geometric morphism) the image of the morphism*

$$\Omega_{Sh(Y)} \stackrel{\Omega_{Sh(X)}!}{\longrightarrow} f_*\Omega_{Sh(X)}$$

under the direct image of the geometric morphism $!^Y : Sh(Y) \to \mathcal{E}$ is $\Omega_{\mathcal{E}}(f) : \Omega_{\mathcal{E}}Y \to \Omega_{\mathcal{E}}X$.

Proof. Given the topos of sheaves, X, $!^X_*\Omega_{Sh(X)} \equiv Sh(X)(1, \Omega_{Sh(X)}) \cong \Omega_{\mathcal{E}}X$.

If f^* is inverse image of the geometric morphism $f : Sh(X) \to Sh(Y)$ then f^* defines a function from the set $\Omega_{\mathcal{E}} Y \cong Sub(1_Y)$ to $\Omega_{\mathcal{E}} X \cong Sub(1_X)$ (since it preserves monomorphisms) which is $\Omega_{\mathcal{E}}(f)$. (Details of this well known fact omitted.) For any morphism $k : A \to B$ in the topos of sheaves over Y, $!_* k : Sh(Y)(1, A) \to Sh(Y)(1, B)$ is given by composition with k. Therefore $!_*(\Omega_X !)$ is a map $\Omega_{\mathcal{E}} Y \cong Sh(Y)(1, \Omega_Y) \to Sh(Y)(1, f_* \Omega_X) \cong Sh(X)(1, \Omega_X)$ where the final isomorphism is adjoint transpose. The result therefore follows from the previous lemma. □

9 Applications: triquotient assignments

The importance of the Joyal and Tierney correspondence is well known. See, for example [JoyTie 84] and [Vermeulen 93] where the correspondence is used to show that open/proper maps are pullback stable and that open/proper surjections are effective descent morphisms. The following immediate consequence is perhaps less well observed.

Theorem 9.1 *Using the standard presentation for locales Y and Z, a presentation for the internal frame $\Omega_{Sh(Y)} Z_p$ (given a locale map $p : Z \to Y$) is*

$$\mathbf{Fr}_{Sh(Y)} \langle !^* L^Z \text{ qua } DL \, | !^* R^Z, \, !^* L^Y \rangle$$

where $!^* L^Y$ is the set of equations given by $!^* \Omega p : !^* L^Y \to !^* L^Z$, $!^* L^Y \stackrel{v}{\to} \Omega_{Sh(Y)} \stackrel{\Omega !}{\to} idl(!^* L^Z)$ where v is the adjoint transpose of $Id : \Omega Y \to \Omega Y \cong !_* \Omega_{Sh(Y)}$. (Here $! : Sh(Y) \to \mathcal{E}$ is the geometric morphism.)

Proof. Recall that in the standard presentation $L^Y \equiv \Omega Y$ etc. and so the presentation makes sense. Since it has been established that the inverse image of a presentation is its pullback and since the pullback of $! : Z \to 1$ along $! : Y \to 1$ is $\pi_1 : Y \times Z \to Y$, it is known that $\mathbf{Fr}_{Sh(Y)} \langle !^* L^Z \text{ qua DL} | !^* R^Z \rangle$ corresponds to $\pi_1 : Y \times Z \to Y$ under $\mathbf{Loc}_{Sh(Y)} \cong \mathbf{Loc}/Y$. Similarly, $\pi_1 : Y \times Y \to Y$ (denoted $(Y \times Y)_{\pi_1}$) corresponds to $\mathbf{Fr}_{Sh(Y)} \langle !^* L^Y \text{ qua DL} | !^* R^Y \rangle$. But, it is straightforward to show that

$$Z_p \stackrel{(p,1)}{\to} (Z \times Y)_{\pi_1} \underset{\Delta \pi_1}{\overset{p \times 1}{\rightrightarrows}} (Y \times Y)_{\pi_1}$$

is an equalizer in $\mathbf{Loc}/Y$, and so is a coequalizer in $\mathbf{Fr}_{Sh(Y)}$. The theorem is simply presenting the coequalizer. $\Delta : Y_{Id} \to (Y \times Y)_{\pi_1}$ is unit of the pullback adjunction between $\mathbf{Loc}$ and $\mathbf{Loc}/Y$ and so the corresponding map $!^* L^Y \stackrel{v}{\to} \Omega_{Sh(Y)}$ is the counit of the adjunction $!^\# \dashv !_*$. □

This result therefore describes arbitrary frames in $Sh(Y)$ in terms of data from frames in $\mathcal{E}$. We now use this description to prove the application that the triquotient maps are pullback stable (originally observed by Vickers, private communication). Firstly, the definition.

Definition 9.2 (Following Vickers, "The double powerlocale and triquotient maps of locales", unpublished note.) A locale map $p : Z \to Y$ has a triquotient assignment if there exists $p_\# : \Omega Z \to \Omega Y$ a dcpo homomorphism such that
(1) $p_\#(c \wedge \Omega p(b)) = (p_\# c \wedge b) \vee p_\#(0)$ and
(2) $p_\#(c \vee \Omega p(b)) = (p_\# c \vee b) \wedge p_\#(1)$.

Notice firstly that the assignment is not in any way unique and secondly that the definition is much weaker than the usual definition of triquotient (e.g. [Plewe 97]) since a locale map with a triquotient assignment need not be surjective, whereas, "triquotient" in the literature invariably means a surjective map. Also note that:

Lemma 9.3 *A dcpo homomorphism $p_\# : \Omega Z \to \Omega Y$ is a triquotient assignment for $p : Z \to Y$ if and only if*

$$p_\#(c_1 \wedge [c_2 \vee \Omega p(b)]) = [p_\# c_1 \wedge b] \vee p_\#(c_1 \wedge c_2) \qquad \text{-}(*)$$

$\forall c_1, c_2 \in \Omega Z$ and $\forall b \in \Omega Y$.

Proof. This proof is an easy algebraic manipulation. If $(*)$ holds then the cases $c_1 = 1$ and $c_2 = 0$ show that (1) and (2) in the definition of a triquotient assignment are satisfied. If $p_\#$ is a triquotient assignment on p then

$$p_\#(c_1 \wedge c_2) \leq p_\#(c_1 \wedge (c_2 \vee \Omega p(b)))$$

and

$$\begin{aligned} p_\#(c_1) \wedge b &\leq [p_\#(c_1) \wedge b] \vee p_\#(0) \\ &= p_\#(c_1 \wedge \Omega p(b)) \quad \text{(since } p_\# \text{ tri., using (1))} \\ &\leq p_\#(c_1 \wedge (c_2 \vee \Omega p(b)), \end{aligned}$$

therefore LHS$\geq$RHS in $(*)$. To complete note that $c_1 \wedge (c_2 \vee \Omega p(b)) \leq (c_1 \wedge c_2) \vee \Omega p(b)$ and so

$$\begin{aligned} p_\#(c_1 \wedge [c_2 \vee \Omega p(b)]) &\leq p_\#(c_1 \wedge [(c_1 \wedge c_2) \vee \Omega p(b)]) \\ &\leq p_\#(c_1) \wedge p_\#((c_1 \wedge c_2) \vee \Omega p(b)) \\ &= p_\#(c_1) \wedge [p_\#(c_1 \wedge c_2) \vee b] \wedge p_\#(1) \\ &= p_\#(c_1) \wedge [p_\#(c_1 \wedge c_2) \vee (b \wedge p_\#(1))] \\ &= p_\#(c_1 \wedge c_2) \vee [p_\#(c_1) \wedge b \wedge p_\#(1)] \\ &= p_\#(c_1 \wedge c_2) \vee [p_\#(c_1) \wedge b]. \end{aligned}$$

□

The next lemma provides a new connection which appears to relate quite closely the class of maps that are effective for descent/pullback stable and the discussions on meet and join stability that are related to the coverage theorems. In particular proof of the lemma hinges on the join and meet stable closure of a frame presentation. It is straightforward to show that given any set of equations

$$\lambda(r) = \bigvee^{\uparrow}\{l' \in L \mid l'\pi r\}$$

(presenting a frame) which are not necessarily join or meet stable we may replace them with all equations of the form

$$c_1 \wedge (c_2 \vee \lambda(r)) = \bigvee^{\uparrow}\{(c_1 \wedge (c_2 \vee l') \mid l' \in L \text{ and } l'\pi r\}$$

over all c_1, c_2 in L. This presents the same frame, but the equations are now meet and join stable. In the particular case (which was true in the last theorem) where the equations are of the form

$$\lambda(r) = \bigvee^{\uparrow}\{0\} \cup \{1 \mid 1_\Omega \leq v(r)\}$$

(recall that $\Omega!(i) = \bigvee^{\uparrow}\{0\} \cup \{1 \mid 1_\Omega \leq i\}$), the meet and join stable closure consists of all the equations of the form

$$c_1 \wedge (c_2 \vee \lambda(r)) = \bigvee^{\uparrow}\{(c_1 \wedge c_2)\} \cup \{(c_1) \mid 1_\Omega \leq v(r)\}.$$

Following a conjecture of Vickers we have:

Lemma 9.4 *There is a 1-1 correspondence between triquotient assignments on $p : Z \to Y$ and internal dcpo homomorphisms,*

$$\mathbf{dcpo}_{Sh(Y)}(\Omega_{Sh(Y)}Z_p, \Omega_{Sh(Y)}).$$

Proof. Let $(L^Z, R^Z, ...)$ be the standard presentation of ΩZ, similarly ΩY, and let, as usual, $! : Sh(Y) \to S1 = \mathcal{E}$ denote the unique geometric morphism. Since we are using standard presentations Ωp is a function from L^Y to L^Z.

By the previous lemma triquotient assignments are exactly monotone maps $n : L^Z \to \Omega Y$ such that n satisfies R^Z and

$$n(c_1 \wedge (c_2 \vee \Omega p(b)) = [n(c_1) \wedge b] \vee n(c_1 \wedge c_2)$$

i.e.

$$L^Z \times L^Z \times L^Y \overset{1\times 1\times \Omega p}{\rightarrow} L^Z \times L^Z \times L^Z \overset{\wedge(\vee\times 1)}{\rightarrow} L^Z \overset{n}{\rightarrow} !_*\Omega_{Sh(Y)} \qquad (\alpha)$$

equals

$$L^Z \times L^Z \times L^Y \overset{(\pi_1,\pi_3,\wedge(\pi_1,\pi_2))}{\rightarrow} L^Z \times L^Y \times L^Z \overset{n\times Id\times n}{\rightarrow}$$
$$!_*\Omega_{Sh(Y)} \times !_*\Omega_{Sh(Y)} \times !_*\Omega_{Sh(Y)} \overset{\vee(\wedge\times 1)}{\rightarrow} !_*\Omega_{Sh(Y)} \qquad (\beta).$$

From the previous theorem: $\Omega_{Sh(Y)}Z_p \cong \mathbf{Fr}_{Sh(Y)}\langle !^*L^Z \text{ qua DL}| !^*R^Z, !^*L^Y\rangle$ and so by the double coverage result $\Omega_{Sh(Y)}Z_p \cong \mathbf{dcpo}_{Sh(Y)}\langle !^*L^Z \text{ qua DL}| !^*R^Z, \overline{!^*L^Y}\rangle$ where $\overline{!^*L^Y}$ is the join and meet stable closure of the relations given by $!^*L^Y$. By the comments in the preamble to this lemma it is therefore clear that dcpo maps $\Omega_{Sh(Y)}Z_p \to \Omega_{Sh(Y)}$ are exactly monotone functions $n' : !^*L^Z \to \Omega_{Sh(Y)}$ which satisfy $!^*R^Z$ and have

$$n'(c_1' \wedge (c_2' \vee [!^*\Omega p](b'))) = \bigvee\nolimits^{\uparrow}_{\Omega_{Sh(Y)}} \{n'(c_1' \wedge c_2')\} \cup \{n'(c_1') \mid 1_\Omega \leq v(b')\}$$

for every c_1' and c_2' in $!^*L^Z$ and every $b' \in !^*L^Y$. The LHS is the adjoint transpose of (α). $\Omega_{Sh(Y)}$ is a frame and so the RHS is equal to:

$$[n'(c_1') \wedge \bigvee\nolimits^{\uparrow} \{0\} \cup \{1 \mid 1_\Omega \leq v(b')\}] \vee n'(c_1' \wedge c_2').$$

But $\bigvee^{\uparrow}\{0\} \cup \{1 \mid 1_\Omega \leq v(b')\}] = v(b')$ (since $\Omega_{Sh(Y)} = P_{Sh(Y)}\{*\}$) and so the result follows since the adjoint transpose of v is the identity. $\Box$

Theorem 9.5 *(Generalizing Plewe, and observed by Vickers.) If $f : X \to Y$ is a locale map and $p : Z \to Y$ has a triquotient assignment then so does the pullback of p along f. Further, the Beck-Chevalley condition is satisfied, i.e.,*

$$\Omega f \circ p_\# = (f^*p)_\# \circ \Omega(p^*f).$$

Proof. Let $p'_\# : \Omega_{Sh(Y)}Z_p \to \Omega_{Sh(Y)}$ be the dcpo map corresponding to the given triquotient assignment $p_\# : \Omega Z \to \Omega Y$ (using the previous result) It has been established that there is an adjunction:

$$\mathbf{dcpo}_{Sh(X)} \underset{f_*}{\overset{f^\#}{\underset{\longrightarrow}{\longleftarrow}}} \mathbf{dcpo}_{Sh(Y)}$$

with $f^\# \dashv f_*$. The following square commutes by naturality of the unit of this adjunction:

$$\begin{array}{ccc} \Omega_{Sh(Y)}Z_p & \overset{\eta_{\Omega_{Sh(Y)}Z_p}}{\longrightarrow} & f_*f^\#\Omega_{Sh(Y)}Z_p \\ p'_\# \downarrow & & \downarrow f_*f^\#p'_\# \\ \Omega_{Sh(Y)} & \overset{\eta_{\Omega_{Sh(Y)}}}{\longrightarrow} & f_*f^\#\Omega_{Sh(Y)} \cong f_*\Omega_{Sh(X)} \end{array}$$

where $f^\#\Omega_{Sh(Y)} \cong \Omega_{Sh(X)}$ since Ω, in any topos, is the free dcpo on the poset $1 + 1 = \{\top \geq \bot\}$. So $f^\#p'_\# : f^\#\Omega_{Sh(Y)}Z_p \to \Omega_{Sh(X)}$ is a dcpo map and so corresponds to a triquotient assignment for f^*p. The Beck-Chevalley condition follows by applying the functor $!^Y_* : Sh(Y) \to S1$ to the naturality square. $\Box$

The pullback stability result for triquotient assignments is now extended to cover surjections.

Lemma 9.6 *Say $p : Z \to Y$ has a triquotient assignment $p_\# : \Omega Z \to \Omega Y$. Then p is a surjection (epi in the category* **Loc***) if $p_\#(1) = 1$ and $p_\#(0) = 0$.*

Proof. p is an epimorphism iff Ωp is monic iff Ωp is an inclusion since the category of frames is suitably algebraic. If $p_\#(1) = 1$ and $p_\#(0) = 0$ then $p_\#\Omega p(b) = b$ by putting $c_1 = 1, c_2 = 0$ in $p_\#(c_1 \wedge [c_2 \vee \Omega p(b)]) = [p_\# c_1 \wedge b] \vee p_\#(c_1 \wedge c_2)$. □

Definition 9.7 $p : Z \to Y$ is a triquotient surjection iff there exists a triquotient assignment with $p_\# : \Omega Z \to \Omega Y$ with $p_\#(1) = 1$ and $p_\#(0) = 0$.

All triquotient surjections are surjections (epimorphisms in **Loc**) but it has not been shown that all surjections with triquotient assignments are triquotient surjections. Our triquotient surjections are exactly Plewe's triquotient maps ([Plewe 97]).

Lemma 9.8 *Triquotient surjections are pullback stable.*

Proof. Immediate from the pullback stability of maps with triquotient assignments and the Beck-Chevalley condition shown above. □

9.1 Proper and open maps

The importance of the notion of triquotient assignment is that it covers the more well known notions of open and proper map. In this subsection we detail how this (known) specialization works. Recall (e.g. [Vermeulen 93], [JoyTie 84]):

Definition 9.9 $p : Z \to Y$ is open if and only if

(i) there exists $\exists_p : \Omega Z \to \Omega Y$ a suplattice homomorphism left adjoint to $\Omega p : \Omega Y \to \Omega Z$

(ii) $\exists_p(c \wedge \Omega p(b)) = b \wedge \exists_p(c)$, for all $c \in \Omega Z, b \in \Omega Y$

and p is proper if and only if

(i) there exists $\forall_p : \Omega Z \to \Omega Y$ a preframe homomorphism right adjoint to $\Omega p : \Omega Y \to \Omega Z$

(ii) $\forall_p(c \vee \Omega p(b)) = b \vee \forall_p(c)$, for all $c \in \Omega Z, b \in \Omega Y$.

(A preframe homomorphism is one which preserves directed joins and finite meets.) The relationship with triquotient assignments on p is easy.

Lemma 9.10 *(i) $p : Z \to Y$ is open iff it has a triquotient assignment, $p_\#$, which is a join semilattice homomorphism, such that $Id \leq \Omega p \circ p_\#$ in the external order on* **Pos**$(\Omega Z, \Omega Z)$.

(ii) $p : Z \to Y$ is proper iff it has a triquotient assignment, $p_\#$, which is a meet semilattice homomorphism, such that $Id \geq \Omega p \circ p_\#$ in the external order on **Pos**$(\Omega Z, \Omega Z)$.

Proof. (i) Say $p_\# : \Omega Z \to \Omega Y$ is (a triquotient assignment and) a join semilattice homomorphism, with $Id \leq \Omega p \circ p_\#$. Certainly condition (ii) in the definition of open holds (i.e. the Frobenius condition) since $p_\#(0) = 0$. By putting $c = 1$ in this condition we see that $p_\# \Omega p(b) \leq b$. But $c \leq \Omega p(p_\#(c))$ for any $c \in \Omega Z$ by assumption and so $p_\# \dashv \Omega p$. Hence p is an open map.

Conversely say p is open, then set $p_\# = \exists_p$. Clearly then (a) $p_\#$ is a dcpo map, (b) $p_\#$ is a join semilattice homomorphism and (c) $Id \leq \Omega p \circ p_\#$. So to finish it must be shown that $p_\#$ satisfies the equation

$$p_\#(c_1 \wedge [c_2 \vee \Omega p(b)]) = [p_\# c_1 \wedge b] \vee p_\#(c_1 \wedge c_2)$$

$\forall c_1, c_2 \in \Omega Z$ and $\forall b \in \Omega Y$. But this is immediate from the (Frobenius) condition (ii) of the definition of open.

(ii) Entirely similar. □

The property of being a meet semilattice homomorphism, we have shown, is pullback stable. (More accurately, we have shown that the property of being a distributive lattice homomorphism is stable, but the proof of this fact amounted to showing that the property of being a semilattice homomorphism is pullback stable. Revisit the proof and discussion surrounding Proposition 7.3.) Certainly we have been clear throughout that the external ordering is preserved by pullback and so the pullback stability results for triquotient assignments given in the previous subsection specialize to:

Theorem 9.11 *Proper and open locale maps are pullback stable.*

Proper/open locale maps are surjections iff $\forall_p(0) = 0/\exists_p(1) = 1$ respectively and so proper/open surjections are pullback stable since the Beck-Chevalley condition holds for triquotient assignments.

10 Main application: The external description of dcpo homomorphisms

This section gives a description of the elements of $\mathbf{dcpo}_{\mathcal{E}}(\Omega_{\mathcal{E}} X, \Omega_{\mathcal{E}} W)$ for any pair of frames in any elementary topos $\mathcal{E}$. This class is equivalent to a class of natural transformations. To do this we will need to discuss the ideal completion of a poset both as a locale and as a topos of presheaves, and the recollection of some basic facts about these representations of P forms the bulk of the first subsection.

10.1 $Idl(P)$ is a locale and a topos

Given a poset, P, the set of monotone maps $P \to \Omega$ is a frame (which is equivalent to the set of upper closed subsets of P). We denote the corresponding locale by $Idl(P)$ since its points are the ideals of P. It is known ([Vickers 93]) and can be easily verified that

$$\begin{aligned}\Omega Idl(P) = \mathbf{Fr}\langle \uparrow p \ (p \in P) | \\ \uparrow p \leq \uparrow q \ (p \geq q) \\ 1 \leq \bigvee_{p \in P} \uparrow p \\ \uparrow p \wedge \uparrow q \leq \bigvee \{\uparrow r | p \leq r, q \leq r\}\rangle\end{aligned}$$

gives a presentation of the frame corresponding to the locale $Idl(P)$. The corresponding topos of sheaves is also denoted $Idl(P)$, and use $h^P : Idl(P) \to \mathcal{E}$ to denote the unique geometric morphism back to the background topos ($\mathcal{E}$). It is well known (for example Theorem 1.6.2 of [Townsend 96] or [Vickers 93]) that locales of this form are constructively spatial. $\Omega Idl(P)$ is isomorphic to the Scott open subsets of $idl(P)$ via

$$\Psi : \Omega Idl(P) \xrightarrow{\cong} \Sigma idl(P)$$
$$a \longmapsto \bigcup_{p \in a} \{I \mid \downarrow p \subseteq I\}$$

where $\Psi^{-1}(U) = \{p \mid \downarrow p \in U\}$. Moreover the continuous maps between them are exactly dcpo maps on the points. In other words,

$$\mathbf{dcpo}_{\mathcal{E}}(idl(L'), idl(L'')) \cong \mathbf{Loc}_{\mathcal{E}}(Idl(L'), Idl(L'')) \cong \mathbf{Top}_{\mathcal{E}}(Idl(L'), Idl(L''))$$

the second equivalence coming from the fact that localic toposes form a full subcategory of the category, **Top**, of toposes and geometric morphisms. Further it will be necessary later to ensure that this correspondence is natural.

Lemma 10.1 *For any posets P and L*

$$\mathbf{Pos}_{\mathcal{E}}(P, \Omega Idl(L)) \cong \mathbf{dcpo}_{\mathcal{E}}(idl(L), \Omega Idl(P))$$

naturally with respect to dcpo maps $h : idl(L') \to idl(L'')$.

Proof. From the definitions, $LHS \cong \mathbf{Pos}_{\mathcal{E}}(P \times L, \Omega)$ and $RHS \cong \mathbf{Pos}_{\mathcal{E}}(L \times P, \Omega)$. A simple calculation using the fact that $[\Omega h(L'' \xrightarrow{a} \Omega)](l') = 1 \iff \exists l'' \in h(\downarrow l') \cap a$ establishes naturality. The mate of $\sigma : P \to \Omega Idl(L')$ under the bijection is $\overline{\sigma} : idl(L') \to \Omega Idl(P)$ where $\overline{\sigma}(I')(p) = 1 \iff \exists l' \in I'$, $\sigma(p)(l') = 1$. Hence

$$\begin{aligned}([(\overline{\Omega h \circ \sigma})(I')](p) &= 1) \iff (\exists l' \in I')(\{[\Omega h \circ \sigma](p)\}(l') = 1) \\ \text{if and only if } (\exists l' &\in I')(\exists l'' \in h(\downarrow l') \cap \sigma(p)) \\ &\iff (\exists l' \in I')([\overline{\sigma}(h(\downarrow l')](p) = 1) \\ &\iff [(\overline{\sigma} \circ h)(I')](p) = 1.\end{aligned}$$

□

As a topos $Idl(P)$ can be described concretely as the collection of all presheaves (i.e. functors) $P \to \mathcal{E}$, with natural transformations as morphisms. The inverse image of the geometric morphism $h^P : Idl(P) \to \mathcal{E}$ sends a set, N, to the constant sheaf (i.e. $a \mapsto N$ for all $a \in P$ and $a \leq a'$ is mapped to the identity function on N). Further it is worth recalling that adjoint transpose of any $k : N \to h^P_* \Omega_{Idl(P)} \cong \Omega Idl(P)$ is the (classifying map of the) subfunctor $k' \subseteq (h^P)^* N$ given by $x \in k'(a)$ iff $k(x)(a) = 1$. The proof of these well known statements follows from the definition of the topos as a collection of sheaves on $Idl(P)$.

So far we have managed "to keep our hands clean" at least to the extent that no proofs have required us to be explicit about the structure of a particular topos in which we are working (though note that the Joyal and Tierney correspondence, in its proof, does require this representation). The next lemma does require this type of explicit representation, but is quite a straightforward result.

Lemma 10.2 *(i) If P is a poset, $q : L \to L'$ a map (i.e. a morphism of $\mathcal{E}$) and I a subpresheaf of $(h^P)^* L$ (i.e. I a subobject of $(h^P)^* L$ in $Idl(P)$) then the presheaf*

$$\begin{aligned} q_\# I \ &: \ P \to Set \\ a &\longmapsto \{x' \in L' \mid \exists x \in I(a),\ x' = q(x)\} \end{aligned}$$

is a subobject of $(h^P)^* L'$ and is the image factorization of $I \hookrightarrow (h^P)^* L \stackrel{(h^P)^* q}{\to} (h^P)^* L'$ in $Idl(P)$.

(ii) If L is also a poset then so is $(h^P)^* L$ and the lower closure (calculated internally in $Idl(P)$) of a subfunctor, I, of $(h^P)^* L$ is calculated pointwise, i.e. $(\downarrow^{IdlP} I)(a) = \downarrow^{\mathcal{E}} I(a)$.

Proof. (i) Image factorization is calculated pointwise in any presheaf category.

(ii) Lower closure is a type of relational composition, which can be expressed via pullback and image factorization. These are done pointwise in a presheaf category. □

We end this subsection with the main equivalence between indexed points of a frame and points of $(h^P)^\#(\Omega_{\mathcal{E}} W)$. It is the naturality of this equivalence that drives the proof of the main application.

Lemma 10.3 *Given a frame $\Omega_{\mathcal{E}} W$, and a poset P (in $\mathcal{E}$) there is a bijection between $\mathbf{dcpo}_{Idl(P)}(1, (h^P)^\#(\Omega_{\mathcal{E}} W))$ and*

(a) monotone maps $P \to \Omega_{\mathcal{E}} W$

(b) dcpo maps $idl(P) \to \Omega_{\mathcal{E}} W$

Before proof, note that for any geometric morphism $g : \mathcal{E}' \to \mathcal{E}$,

$$\mathbf{dcpo}_{\mathcal{E}'}(1, (g')^{\#}(\Omega_{\mathcal{E}} W)) \cong \mathcal{E}'(1, (g')^{\#}(\Omega_{\mathcal{E}} W)),$$

but we would like to keep the order enrichment in mind and so use the former notation.

Proof. Say that $\Omega_{\mathcal{E}} W$ is presented by the distributive lattice L^W subject to the relations R^W. Then $(h^P)^{\#}(\Omega_{\mathcal{E}} W)$ is presented by $(h^P)^* L^W$ subject to $(h^P)^* R^W$. Now, $\mathbf{dcpo}_{Idl(P)}(1, (h^P)^{\#}(\Omega_{\mathcal{E}} W))$ is exactly the collection (i.e. external homset/class) of global elements of $(h^P)^{\#}(\Omega_{\mathcal{E}} W)$, i.e. of maps $1 \to (h^P)^{\#}(\Omega_{\mathcal{E}} W)$ which we known (by the explicit description of the frame in terms of C-ideals) to be exactly the join semilattice homomorphisms $(h^P)^* L^W \to (\Omega_{Idl(P)})^{op}$ which satisfy $(h^P)^* R^W$. By taking the adjoint transpose and applying Theorem 6.1 it is clear that this SET is of join semilattice homomorphisms $L^W \to ((h^P_*)(\Omega_{Idl(P)}))^{op}$ which satisfy R^W. By definition of $Idl(P)$, $(h^P_*)(\Omega_{Idl(P)})$ is the set of monotone maps $P \to \Omega$, and since the intersection of upper closed subsets is upper closed, the meet operation on $P \to \Omega$ is calculated pointwise. If $\sigma_I : L^W \to ((h^P_*)(\Omega_{Idl(P)}))^{op}$ satisfies R^W then for every $r \in R^W$

$$\sigma_I(\lambda(r)) = \bigwedge\nolimits_{\Omega_{\mathcal{E}} Idl(P)} \{\sigma_I(l) \mid l \pi r\}$$

and so, for any $p \in P$,

$$\sigma_I(\lambda(r))(p) = \bigwedge\nolimits_{\Omega_{\mathcal{E}}} \{\sigma_I(l)(p) \mid l \pi r\}.$$

This says that the (double) exponential transpose of $\sigma_I, \ulcorner \sigma_I \urcorner : P \to (L^W \to \Omega_{\mathcal{E}}^{op})$ has the property that $\ulcorner \sigma_I \urcorner (p, \_) : L^W \to \Omega^{op}$ satisfies R^W. That the property of being a join semilattice is preserved by the adjoint transpose followed by the exponential transpose follows similarly and so $\ulcorner \sigma_I \urcorner$ factors through $\Omega_{\mathcal{E}} W$ since this is a collection of C-ideals. That $\ulcorner \sigma_I \urcorner$ is monotone follows since σ_I is an indexed collection of monotone maps.

The correspondence between monotone maps and dcpo maps ((a) and (b)) is immediate since $idl(L)$ (the set of ideals, as opposed to the locale) is the free dcpo qua poset. □

10.2 The functor $\Lambda^{\Omega_{\mathcal{E}} W} : (\mathbf{Top}/\mathcal{E})^{op} \to SET$

Our description of $\mathbf{dcpo}_{\mathcal{E}}(\Omega_{\mathcal{E}} X, \Omega_{\mathcal{E}} W)$ will be as the set of natural transformations from $\Lambda^{\Omega_{\mathcal{E}} X}$ to $\Lambda^{\Omega_{\mathcal{E}} W}$. These functors must be defined.

Definition 10.4 $\Lambda^{\Omega_{\mathcal{E}} W} : (\mathbf{Top}/\mathcal{E})^{op} \to SET$, takes the object $g' : \mathcal{E}' \to \mathcal{E}$ (of $\mathbf{Top}/\mathcal{E}$) to $\mathbf{dcpo}_{\mathcal{E}'}(1, (g')^{\#}(\Omega_{\mathcal{E}} W))$. Since $h^{\#}(1) = 1$, for $h : \mathcal{E}' \to \mathcal{E}''$ a morphism of $\mathbf{Top}/\mathcal{E}$, $[\Lambda^{\Omega_{\mathcal{E}} W}(h)](k) = h^{\#}(k)$ is well defined.

Since $(g')^\#$ is, when applied to locales, pullback, another way of looking at this functor is that $\Lambda^{\Omega_\mathcal{E} W}(g')$ = the opens of $Sh(W) \times_\mathcal{E} \mathcal{E}'$, i.e. of the pullback of the localic geometric morphism $Sh(W) \to \mathcal{E}$ along g'. So, as in the Introduction, the functors are

$$\Lambda^{\Omega_\mathcal{E} W} : (\mathbf{Top}/\mathcal{E})^{op} \to SET$$
$$(h : \mathcal{E}' \to \mathcal{E}) \longmapsto \mathbf{Top}(\mathcal{E}' \times_\mathcal{E} Sh(W), Sh(\mathbb{S}))$$

where $\mathbb{S}$ is the Sierpiński locale, i.e. the locale whose frame is the free frame on the singleton set 1. However we shall not rely on this description until the final applications below, and then only in the context of locales. Thus, for future work, it is clear that we are looking at the categories $\mathcal{E}' \times_\mathcal{E} Sh(W)$ as indexed carriers for the data of dcpo maps. But since we can prove the main result without this interpretation I am suppressing it.

The definition (on morphisms) also has an alternative characterization which will be used in the proofs.

Lemma 10.5 *(Alternative characterization of $\Lambda^{\Omega_\mathcal{E} W}$ on morphisms). As an action on C-ideals $[\Lambda^{\Omega_\mathcal{E} W}(h)](I) = \bigvee_{(g')^\# \Omega_\mathcal{E} W} h^* I$ if $h : \mathcal{E}' \to \mathcal{E}''$ is a morphism in $\mathbf{Top}/\mathcal{E}$.*

Proof. It needs to be checked that for any map $n_I : 1 \to (g'')^\# \Omega_\mathcal{E} W$ corresponding to a C-ideal I of $(g'')^\# \Omega_\mathcal{E} W$, $h^\#(n_I)$ corresponds to the open $\bigvee_{(g')^\# \Omega_\mathcal{E} W} h^* I$ of $(g')^\# \Omega_\mathcal{E} W$. Now $h^\#(n_I)$ is defined by its adjoint transpose: $h^\#(n_I)$ is the adjoint transpose of the composition

$$1 \xrightarrow{n_I} (g'')^\# \Omega_\mathcal{E} W \xrightarrow{\eta} h_* h^\# (g'')^\# \Omega_\mathcal{E} W$$

where η is the unit of the adjunction $h^\# \dashv h_*$. η is the unique frame homomorphism that extends the map $e : (g'')^* L^W \to h_* h^\# (g'')^\# \Omega_\mathcal{E} W$ which is the adjoint transpose of the universal map of the generators $h^*(g'')^* L^W \to h^\#(g'')^\# \Omega_\mathcal{E} W$. Therefore, as the C-ideal I is the join of the indexing $I \subseteq (g'')^* L^W \to (g'')^\# \Omega_\mathcal{E} W$, and η preserves joins, $\eta \circ n_I(*)$ is equal to the join (in $h_* h^\#(g'')^\# \Omega_\mathcal{E} W$) of the indexing map $\alpha : I \to (g'')^* L^W \xrightarrow{e} h_* h^\#(g'')^\# \Omega_\mathcal{E} W$. By definition of join in $h_* h^\#(g'')^\# \Omega_\mathcal{E} W$, $\eta \circ n_I(*)$ is therefore equal to the adjoint transpose of the join of the adjoint transpose of α. But the adjoint transpose α is $h^* I \to (g')^\# \Omega_\mathcal{E} W$ and so the result follows. □

10.3 Naturality lemma and 'weak exponential' lemma

Lemma 10.6 *(Naturality lemma) The bijection between dcpo maps $idl(L) \to \Omega_\mathcal{E} W$ and $\mathbf{dcpo}_{Idl(L)}(1, (h^L)^\#(\Omega_\mathcal{E} W))$ is natural with respect to dcpo maps $h : idl(L') \to idl(L'')$ and with respect to dcpo maps $q : \Omega_\mathcal{E} X \to \Omega_\mathcal{E} W$.*

Proof. Say I is a C-ideal of $(h^{L''})^\#(\Omega_\mathcal{E} W)$ (in the topos of sheaves $Idl(L'')$). $\Lambda^{\Omega_\mathcal{E} W}(h)(I)$ is, by the last lemma, the C-ideal generated by $h^* I$, which is classified by the map $v \circ h^* \chi_I$ where v is the adjoint transpose of the unique frame homomorphism (in $Idl(L'')$) from $\Omega_{Idl(L'')} \to h_* \Omega_{Idl(L')}$. The proof will be completed provided we can show that this subset is equivalent to $idl(L') \xrightarrow{h} idl(L'') \xrightarrow{z_I} \Omega_{\mathcal{E}} W$ under the bijection between dcpo maps $idl(L') \to \Omega_{\mathcal{E}} W$ and $\mathbf{dcpo}_{Idl(P)}(1, (h^{L'})^{\#}(\Omega_{\mathcal{E}} W))$, where z_I is the image of I under this same bijection applied to L''. If this can be shown then $h^* I$ will be a C-ideal, and the naturality is established.

To establish this claim, notice that the bijection of Lemma 10.3 is essentially the process of taking the adjoint transpose (followed by the exponential transpose) The adjoint transpose $v \circ h^* \chi_I$ via the geometric morphism $h^{L'}$ is found by taking the adjoint transpose with respect to h and then with respect to $h^{L''}$ (since $h \circ h^{L''} = h^{L'}$). The first adjoint transpose is

$$(h^{L''})^* L^W \xrightarrow{\chi_I} \Omega_{Idl(L'')} \xrightarrow{\Omega_{Idl(L'')}!} h_* \Omega_{Idl(L')}$$

and the second is

$$L^W \xrightarrow{\overline{I}} \Omega_{\mathcal{E}} Idl(L'') \xrightarrow{\Omega_{\mathcal{E}} h} \Omega_{\mathcal{E}} Idl(L')$$

where $\overline{I}$ is the adjoint transpose of I. Now Lemma 10.1 with $(L^W)^{op}$in the place of P shows us that the exponential transpose of the monotone map $(L^W)^{op} \xrightarrow{\overline{I}} \Omega_{\mathcal{E}} Idl(L'') \xrightarrow{\Omega_{\mathcal{E}} h} \Omega_{\mathcal{E}} Idl(L')$ is $idl(L') \xrightarrow{h} idl(L'') \xrightarrow{z_I} \Omega_{\mathcal{E}} W$ as required.

For the second claim of naturality, say $q : \Omega_{\mathcal{E}} X \to \Omega_{\mathcal{E}} W$ and I is a C-ideal of $(h^L)^{\#}(\Omega_{\mathcal{E}} X)$. Then (use e.g. Theorem 6.2), $(h^L)^{\#}(q)(I)$ is the C-ideal generated by the image factorization of

$$I \hookrightarrow (h^L)^* L^X \xrightarrow{(h^L)^*(\overline{q})} (h^L)^* L^W \qquad (*)$$

(where $\overline{q} : L^X \to L^Y$ is using the canonical presentations, i.e. $\overline{q} = q$, $\Omega_{\mathcal{E}} X = L^X$ etc.). It must be shown that this is the image under the bijection $\mathbf{dcpo}_{\mathcal{E}}(idl(L), \Omega_{\mathcal{E}} W) \cong \mathbf{dcpo}_{Idl(L)}(1, (h^L)^{\#}(\Omega_{\mathcal{E}} W))$ of

$$idl(L) \xrightarrow{z_I} \Omega_{\mathcal{E}} X \xrightarrow{q} \Omega_{\mathcal{E}} W.$$

This last function, as an L indexed collection of subsets of L^W is given by $\{c \in L^W \mid c \le q(z_I \downarrow l)\}$ for $l \in L$ (recall that all C-ideals are principal in the standard presentation). The image factorization of $(*)$, we have shown in Lemma 10.2, is the L indexed collection of subsets $\{c \in L^W \mid \exists a \le z_I \downarrow l, c = \overline{q}(a)\}$. We must show that the C-ideal closure of this subfunctor of $(h^L)^* L^W$ is given by the indexing $\{c \in L^W \mid c \le q(z_I \downarrow l)\}$, and for this it is clearly sufficient to simply show that the lower closure of the subfunctor is given by this indexing. This is true because lower closure is calculated pointwise, as we have mentioned explicitly in part (ii) of Lemma 10.2. □

The next lemma, interpreted for locales, indicates that the function space $\mathbb{S}^X$ exists weakly in the category **Loc** where $\mathbb{S}$ is the Sierpiński locale (see

[TowVic 02]). It is offered here only as a technical step (though perhaps we conjecture that $[Set]^{\mathcal{E}}$ exists weakly in the category of Grothendieck toposes, for any topos Grothendieck topos $\mathcal{E}$, where $[Set]$ is the object classifier).

Lemma 10.7 *If $g : \mathcal{E}' \to \mathcal{E}$ is a geometric morphism, $\Omega_{\mathcal{E}}X$ is a frame in $\mathcal{E}$ and $I \in \mathbf{dcpo}_{\mathcal{E}'}(1, g^{\#}\Omega_{\mathcal{E}}X)$, then there exists a geometric morphism $h : \mathcal{E}' \to Idl(L^X)$ such that $I = \Lambda^{\Omega_{\mathcal{E}}X}(h)(I_{ev})$ where I_{ev} is the element (C-ideal) of $\mathbf{dcpo}_{Idl(L^X)}(1, (h^{L^X})^{\#}\Omega_{\mathcal{E}}X)$ corresponding to the universal map of generators $L^X \to \Omega_{\mathcal{E}}X$.*

Proof. By the hyperconnected localic factorization (see e.g. A4.6 of [Johnstone 02]) g is the composition $\mathcal{E}' \xrightarrow{l} Y \to \mathcal{E}$ where Y is the locale determined by the frame $g_*\Omega_{\mathcal{E}'}$. If I is a point of $g^{\#}\Omega_{\mathcal{E}}X$ then I is an internal C-ideal and therefore corresponds to a map $g^*L^X \to \Omega_{\mathcal{E}}^{op}$ which is a join semilattice homomorphism. This therefore gives rise to a join semilattice homomorphism $L^X \to g_*\Omega_{\mathcal{E}}^{op}$ which, it can then be checked, satisfies the universal frame theoretic characterization of $\Omega Idl(L^X)$ given at the beginning of this section. Hence there is a geometric morphism $Y \xrightarrow{k} Idl(L^X)$, and so there exists $h : \mathcal{E}' \to Idl(L^X)$ given by $h = k \circ l$. Notice that $h^{L^X} \circ h = g$. Now $\Lambda^{\Omega_{\mathcal{E}}X}(h)(I_{ev})$ is the C-ideal generated by h^*I_{ev}. The classifying map of h^*I_{ev} is

$$g^*L^X \stackrel{h^*\chi_{I_{ev}}}{\longrightarrow} h^*\Omega_{Idl(L^X)} \stackrel{v}{\longrightarrow} \Omega_{\mathcal{E}'} \qquad (*)$$

where v is the adjoint transpose (via h) of $\Omega_{Idl(L^X)} \stackrel{\Omega_{Idl(L^X)}(!)}{\longrightarrow} h_*\Omega_{\mathcal{E}'}$. Since $h^{L^X} \circ h = g$, the adjoint transpose of this map (via g) can be found by taking the adjoint transpose via h and then via h^{L^X}. The proof will then be complete provided that these adjoint transposes correspond to the adjoint transpose of I via g.

Firstly, the adjoint transpose of $(*)$ via h is

$$(h^{L^X})^*L^X \stackrel{\chi_{I_{ev}}}{\longrightarrow} \Omega_{Idl(L^X)} \stackrel{\Omega_{Idl(L^X)}(!)}{\longrightarrow} h_*\Omega_{\mathcal{E}'}$$

and the adjoint transpose of this via h^{L^X} is

$$L^X \stackrel{\uparrow}{\longrightarrow} \Omega Idl(L^X) \stackrel{\Omega_{\mathcal{E}}k}{\longrightarrow} g_*\Omega_{\mathcal{E}'}$$

where the map denoted "↑" is the inclusion of generators by definition of I_{ev} and the second map is $\Omega_{\mathcal{E}}k$ since $\Omega_{\mathcal{E}}k = (h^{L^X})_*\Omega_{Idl(L^x)}(!)$ by Lemma 8.4. This composition is, by the definition of $\Omega_{\mathcal{E}}k$ via the universal frame theoretic characterization of $\Omega Idl(L^X)$, equal to the adjoint transpose of I via g as required. □

10.4 Main application

The proof of the main application is now relatively easy.

Theorem 10.8 *For any two locales X, W in a topos $\mathcal{E}$, there is a bijection between dcpo homomorphisms $q : \Omega_{\mathcal{E}}X \to \Omega_{\mathcal{E}}W$ and natural transformations $\Lambda^{\Omega_{\mathcal{E}}X} \dot{\to} \Lambda^{\Omega_{\mathcal{E}}W}$. This bijection is natural in frame homomorphisms $\Omega_{\mathcal{E}}W \to \Omega_{\mathcal{E}}W'$.*

Proof. Firstly any dcpo map $q : \Omega_{\mathcal{E}}X \to \Omega_{\mathcal{E}}W$ gives rise to a natural transformation $\Lambda^{\Omega_{\mathcal{E}}X} \dot{\to} \Lambda^{\Omega_{\mathcal{E}}W}$. Recall that for any $g' : \mathcal{E}' \to \mathcal{E}$

$$\Lambda^{\Omega_{\mathcal{E}}X}(g') = \mathbf{dcpo}_{\mathcal{E}'}(1, (g')^{\#}\Omega_{\mathcal{E}}X)$$

and so by defining $\alpha^q_{g'}(n) = (g')^{\#}(q) \circ n$ a natural transformation is obtained since (given $l : \mathcal{E}'' \to \mathcal{E}'$ in $\mathbf{Top}/\mathcal{E}$)

$$\begin{array}{ccc} \mathbf{dcpo}_{\mathcal{E}'}(1, (g')^{\#}\Omega_{\mathcal{E}}X) & \overset{\alpha^q_{g'}}{\to} & \mathbf{dcpo}_{\mathcal{E}'}(1, (g')^{\#}\Omega_{\mathcal{E}}W) \\ l^{\#} \downarrow & & \downarrow l^{\#} \\ \mathbf{dcpo}_{\mathcal{E}''}(1, (g'')^{\#}\Omega_{\mathcal{E}}X) & \overset{\alpha^q_{g''}}{\to} & \mathbf{dcpo}_{\mathcal{E}''}(1, (g'')^{\#}\Omega_{\mathcal{E}}W) \end{array}$$

commutes.

On the other hand given a natural transformation $\alpha : \Lambda^{\Omega_{\mathcal{E}}X} \dot{\to} \Lambda^{\Omega_{\mathcal{E}}W}$, the monotone map (from L^X to $\Omega_{\mathcal{E}}W$) corresponding $\alpha_{h^{L^X}}(I_{ev})$ satisfies the relations R^X. To see this notice that $\Lambda^{\Omega_{\mathcal{E}}X}(e_1)(I_{ev}) = \Lambda^{\Omega_{\mathcal{E}}X}(e_2)(I_{ev})$ where $e_1, e_2 : Idl(R^X) \to Idl(L^X)$ are the geometric morphisms implied by the presentation (treating R^X as a discrete poset); this is true since the universal map of generators certainly satisfies the relations (and we are applying the first part of the naturality lemma). Hence by applying the naturality of α at e_1, e_2 it can be seen that $\alpha_{h^{L^X}}(I_{ev})$ satisfies the relations (again using the first part of the naturality lemma). It therefore gives rise to a dcpo map $q_\alpha : \Omega_{\mathcal{E}}X \to \Omega_{\mathcal{E}}W$. Given any $q : \Omega_{\mathcal{E}}X \to \Omega_{\mathcal{E}}W$, q_{α^q} is then determined by $\alpha^q_{h^{L^X}}(I_{ev}) = (h^{L^X})^{\#}(q) \circ I_{ev}$ and this corresponds to $L^X \to \Omega_{\mathcal{E}}X \overset{q}{\to} \Omega_{\mathcal{E}}W$ by the second part of the naturality lemma above, i.e. $q = q_{\alpha^q}$.

Notice that for any $I \in \mathbf{dcpo}_{\mathcal{E}}(1, g^{\#}\Omega_{\mathcal{E}}X)$ (any $g : \mathcal{E}' \to \mathcal{E}$) the previous lemma shows that $I = \Lambda^{\Omega_{\mathcal{E}}X}(h)(I_{ev})$ for some $h : \mathcal{E}' \to Idl(L^X)$ and so by naturality every natural transformation $\Lambda^{\Omega_{\mathcal{E}}X} \dot{\to} \Lambda^{\Omega_{\mathcal{E}}W}$ is uniquely determined by $\alpha_{h^{L^X}}(I_{ev})$. So, given any α, α^{q_α} evaluated at I_{ev} (at stage h^{L^X}) is

$$(h^{L^X})^{\#}(q_\alpha) \circ I_{ev} = L^X \to \Omega_{\mathcal{E}}X \overset{q_\alpha}{\to} \Omega_{\mathcal{E}}W = \alpha_{h^{L^X}}(I_{ev})$$

where the first equality is by the second part of the naturality lemma and the second is the from the definition of q_α. It follows that $\alpha = \alpha^{q_\alpha}$ and a bijection is established.

Naturality of this bijection is immediate from the definition of α^q given and the fact that $(g')^{\#}$ is a functor for any $g' : \mathcal{E}' \to \mathcal{E}$. □

This theorem specializes to the main technical insight of [TowVic 02].

Theorem 10.9 *Fix a locale X. There exist bijections*

$$\Phi_W : Nat(\mathbf{Loc}(\_\times X, \mathbb{S}), \mathbf{Loc}(\_\times W, \mathbb{S})) \rightarrow \mathbf{dcpo}(\Omega X, \Omega W)$$

natural in locales W. Here, $\mathbf{Loc}(\_\times X, \mathbb{S}), \mathbf{Loc}(\_\times W, \mathbb{S})$ *are functors* $\mathbf{Loc}^{op} \rightarrow \mathbf{Set}$.

Proof. From the Joyal and Tierney correspondence it is clear that $\mathbf{Loc}(\_\times X, \mathbb{S})$ is the same as $\Lambda^{\Omega X}$ restricted to **Loc**. But the ideal completion locale, as a topos, is localic. Therefore the whole proof given above can be carried out looking only at localic toposes. □

The main result of [TowVic 02] can be recovered.

Corollary 10.10 *If X is a locale then the exponential $\mathbb{S}^{\mathbb{S}^X}$ exists in* $[\mathbf{Loc}^{op}, \mathbf{Set}]$ *and is naturally isomorphic to the representable functor* $\mathbf{Loc}(\_, \mathbb{P}X)$.

Here $\mathbb{P}X$ is the double power locale on X. It can be found by composing, in either order, the upper power locale functor followed by the lower power locale functor. By definition $\Omega\mathbb{P}X = \mathbf{Fr}\langle\Omega X$ qua dcpo$\rangle$, see [JoVic 91].

Proof. This is a question of unravelling the definitions since the general points of $\mathbb{P}X$ (i.e. locale maps $Y \rightarrow \mathbb{P}X$) are exactly the dcpo maps between frames with domain ΩX. It is straight forward to verify that $\mathbb{S}^X$ exists in $[\mathbf{Loc}^{op}, \mathbf{Set}]$ since it is given by the functor $\mathbf{Loc}(\_\times X, \mathbb{S})$. □

In a manner very similar to the manner by which triquotient results can be used to show results about proper and open maps, this last corollary specializes to results about the points of the upper and lower power locales. Details are available in [TowVic 02].

Therefore an entirely external characterization of the general points of the double power locale is available. This means that a categorical axiomatization of a double power space functor could be made for any order enriched category with a (suitably axiomatized) Sierpiński space. This would be the subject of further work, in line with a project (advocated by Vickers, private communication) of re-expressing topology in terms geometric reasoning.

11 Concluding comments

This paper concentrates on the technical observation that dcpo presentations present and that since the presentations for dcpos are stable under the inverse image of geometric morphisms, one is able to define a left adjoint to $f_* : \mathbf{dcpo}_{\mathcal{E}} \rightarrow \mathbf{dcpo}_{\mathcal{E}'}$. Further, from the double coverage result, locale maps can be described as particular dcpo maps and so this left adjoint is locale pullback. This ability to move the dcpo maps from one topos to another (via the adjunction of the geometric morphism) allows the usual applications to go through, e.g. triquotient surjections are pullback stable, covering the same results for open and proper surjections. Moreover the form of the definition of triquotient (and hence of proper and open) appears as a natural consequence of the presentation (as a dcpo) of a locale internal in a topos of sheaves. It is shown (following a conjecture of Vickers) that the weak triquotient assignments of $p : Z \to Y$ (i.e. those maps used to define triquotient/proper/open) are exactly the global points of the double power locale of Z, viewed as a locale internal to sheaves over Y.

The main application here is to use this technology to give a topos theoretic version of the result of Townsend/Vickers which describes the points of the double power locale as certain natural transformations (therefore, in effect, giving an external characterization of the notion of Scott continuity of maps between internal frames).

By clearly separating out the infinitary directed join structure of frames from the finitary (distributive lattice) structure it is hoped that further light is shed on the parallel that exists between proper and open (e.g. [Townsend 96]), since the basis of that parallel appears to be that the finitary structure is dualised whilst the infinitary structure remains fixed.

Another invisible motivation for the work has been an attempt to answer the question of what the topos theoretic analogue to the upper power locale should be (see B4.5 of [Johnstone 02] and [Karazeris 01])? More broadly, and indeed more simply: What is the topos theoretic version of a dcpo presentation for a Grothendieck topos? It is hoped that the work offered here, in its structure at least, will have a topos theoretic version (with toposes in the place of locales). Notably, if a topos theoretic version of the subsection "Shortcut to the Applications" is available, then it should be possible to replicate the main application for toposes. We therefore end with a

Conjecture 11.1 *There is a 1-1 correspondence between filtered cocontinuous functors between Grothendieck toposes, $\mathcal{E}, \mathcal{E}'$ and natural transformations $\Lambda^{\mathcal{E}} \stackrel{\cdot}{\to} \Lambda^{\mathcal{E}'}$, where $\Lambda^{\mathcal{E}}$ is the functor*

$$\Lambda^{\mathcal{E}} : (\mathbf{BTop}/Set)^{op} \to SET$$
$$(h : \mathcal{E}' \to Set) \longmapsto \mathbf{BTop}/Set(\mathcal{E}' \times_{Set} \mathcal{E}, [Set])$$

and $[Set]$ is the object classifier.

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This paper is dedicated to Malcolm Townsend, on the occasion of his sixty-fifth birthday.

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