AXIOMATIC CHARACTERIZATION OF THE CATEGORY OF LOCALES DRAFT

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Abstract. An axiomatic characterization of the category of locales is given. The main idea is to introduce the ideal completion locale as a categorical tensor with the terminal object. The Sierpiński locale is then the ideal completion of **2**. The spectral locales are axiomatised as abstract logarithms base Sierpiński (following a suggestion by Vickers).

The axiomatization is not elementary since the category is required to be enriched over directed complete partial orders.

1. Introduction

In this paper we introduce 8 axioms on a category $\mathbf{C}$. It is shown that the category $\mathbf{Loc}$, of locales, satisfies the axioms and that for any other category, $\mathbf{C}$, if $\mathbf{C}$ satisfies the axioms then $\mathbf{C} \simeq \mathbf{Loc}$. Therefore the axioms characterize the category of locales. The category of locales is, by definition, the opposite of the category of frames.

The axioms on $\mathbf{C}$ state that,

Axiom 1. *it is enriched over directed complete partial orders,*

Axiom 2. *it is cartesian,*

Axiom 3. *it has aribtrary tensors with posets,*

Axiom 4. *the tensor commutes with product via* $(P \otimes X) \times Y \cong P \otimes (X \times Y)$ *for any poset* P *and objects* X *and* Y*,*

Axiom 5. *for any poset* P*,* $\mathbf{C}(1, Idl(P)) \cong idl(P)$ *where* $Idl(P)$ *is the tensor of* P *with* 1 *and* $idl(P)$ *is the set of ideals of* P*,*

Axiom 6. *given objects* X *and* Y*, any frame homomorphism* $\mathbf{C}(Y, \mathbb{S}) \to \mathbf{C}(X, \mathbb{S})$ *is uniquely of the form* $\mathbf{C}(f, \mathbb{S})$ *for* $f : X \to Y$ *where* $\mathbb{S}$ *is* $Idl(\mathbf{2})$*,*

Axiom 7. *given objects* X *and* Y*, the function* $(f : X \to Y) \longmapsto \mathbf{C}(f, \mathbb{S}) : \mathbf{C}(Y, \mathbb{S}) \to \mathbf{C}(X, \mathbb{S})$ *takes equalizers to frame coequalizers, and*

Axiom 8. *for every distributive lattice,* L*, there is an object* $Spec(L)$ *of* $\mathbf{C}$ *with the property that* $\mathbf{C}(Spec(L), \mathbb{S}) \cong idl(L)$*.*

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Whilst it is interesting to note that an axiomatic account can be given for the category of locales, it must be cautioned that these axioms are not elementary since they use the full force of an external power set. For example, this is needed to describe the theory of directed complete partial orders required for Axiom 1. The reader is further cautioned that it is not clear whether this work is simply exposing a categorical triviality.

2. Locales

For background on the category of locales consult [Joh 82] or [Vic 89]. The idea is that the category of locales behaves like the category of topological spaces and so provides an environment in which to do topology. Locales are defined in terms of frames. A frame is a complete lattice, A, such that for every subset $T \subseteq A$ and every element $a \in A$,

$$a \wedge \bigvee T = \bigvee \{a \wedge t \mid t \in T\}$$

where $\bigvee$ and $\wedge$ are the lattice theoretic join and meet operations. A frame homomorphism preserves arbitrary joins and finite meets and so a category, **Fr**, of frames is defined. The category **Loc** of locales is defined by

$$\mathbf{Loc} \equiv \mathbf{Fr}^{op}.$$

The notation is that given a locale, X, there is the *corresponding frame* ΩX. The difference between a locale and a frame is therefore only notational. On maps, given a locale map, $f : X \to Y$, there is the corresponding frame homomorphism $\Omega f : \Omega Y \to \Omega X$. A locale map is the localic model of a continuous map between topological spaces, but for this paper we will not elaborate further on the topological connection.

The category of locales is, by construction, locally small and the homsets are clearly posets using the pointwise ordering; that is $f_1 \sqsubseteq f_2 : X \to Y$ iff $\Omega f_1(b) \leq_{\Omega X} \Omega f_2(b)$ for every $b \in \Omega Y$. It is easy to verify that the category of locales is enriched over directed complete partial orders (dcpos); consult, for example, Lemma 1.11 in [Joh 82]. Further, for example by II, 2.11 and 2.12 in [Joh 82], we have that the category of locales is cartesian (has finite limits). The category of locales therefore satisfies Axioms 1 and 2.

For any poset L, we can form $idl(L)$, the set of ideals of L. An ideal is a lower closed and directed subset of L. If further L is a distributive lattice then $idl(L)$ can be verified to be the free frame on the distributive lattice L; that is, $idl(\_)$ is left adjoint to the forgetful functor $\mathbf{Fr} \to \mathbf{DLat}$ where **DLat** is the category of distributive lattices (e.g. Corollary II, 2.11 in [Joh 82]). Since $idl(L)$ is a frame for L a distributive lattice we can define a locale $Spec(L)$ by $\Omega Spec(L) \equiv idl(L)$.

In fact every frame can be described as a coequalizer of frames of the form $idl(L)$ with L a distributive lattice, and this will be key to our proof.

Lemma 1. *For any frame, ΩX, there exist distributive lattices L and $\overline{R}$ and frame homomorphisms $\Omega e_1, \Omega e_2 : idl(\overline{R}) \to idl(L)$ and $\Omega i : idl(L) \twoheadrightarrow \Omega X$ such that*

$$idl(\overline{R}) \underset{\Omega e_2}{\overset{\Omega e_1}{\rightrightarrows}} idl(L) \overset{\Omega i}{\twoheadrightarrow} \Omega X$$

is a coequalizer in the category of frames.

Proof. Take L to be the ΩX and take R to be the set $idl(\Omega X)$ (i.e. forget the ordering on the ideals, R is discrete as a poset). Define two maps $\epsilon_1, \epsilon_2 : R \to idl(L)$ by $\epsilon_1(I) = I$ (i.e. the identity) and $\epsilon_2(I) = \downarrow \vee^{\uparrow} I$, i.e. the lower closure of $\{\vee^{\uparrow} I\}$ where $\vee^{\uparrow} I$ is the (directed) join of the elements of the ideal I. It can be checked that an arbitrary frame homomorphism $\Omega f : \Omega X \to \Omega W$ is exactly a distributive lattice homomorphism $\Omega f : \Omega X \to \Omega W$ such that its extension $\widetilde{\Omega f} : idl(\Omega X) \to \Omega W$ to a frame homomorphism, composes equally with ϵ_1 and ϵ_2.

To complete the proof take $\overline{R}$ to be the free distributive lattice on R (see I, 4.8 [Joh 82] for a construction) and let $\Omega e_1, \Omega e_2$ be the extensions (to distributive lattice homomorphisms, then to frame homomorphisms) of ϵ_1, ϵ_2. ☐

3. Categorical Tensors

Given an object, A, of a 2-category, $\mathfrak{K}$, then for any small category $\mathcal{C}$, the tensor $\mathcal{C} \otimes A$ is defined as the (weighted) colimit which enjoys

$$[\mathcal{C}, \mathfrak{K}(A, B)] \simeq \mathfrak{K}(\mathcal{C} \otimes A, B)$$

for any object B of $\mathfrak{K}$, where the equivalence is defined by composition with a functor $\mathcal{C} \to \mathfrak{K}(A, \mathcal{C} \otimes A)$. (Here $[\_, \_]$ denotes the class of functors.) See Example B1.1.4(b) in [Joh 02] for this definition.

Our context is not of 2-cateogeries, but only order-enriched categories. The existence of tensors, stipulated by Axiom 3, is therefore that for any poset P and any object X of $\mathbf{C}$ there exists an object $P \otimes X$ of $\mathbf{C}$ and a monotone map $\theta^{P,X} : P \to \mathbf{C}(X, P \otimes X)$ such that composition with this maps induces an order-isomorphism,

$$[P, \mathbf{C}(X, Y)] \cong \mathbf{C}(P \otimes X, Y)$$

for any Y.

Lemma 2. *Axioms 3, 4 and 5 are true with* $\mathbf{C} = \mathbf{Loc}$.

This is well known of locales, e.g. [TowVic 03]. In fact the 2-categorical statement of Axiom 3 is known for the larger category of Grothendieck toposes, e.g. B3.4.7 in [Joh 02].

Proof. Axiom 3. Given a poset P, if ΩX^P denotes the set of monotone maps from P to ΩX then, it can be verified, ΩX^P is a frame. Join and meet are calculated pointwise, and from this it is clear that ΩX^P is the tensor $P \otimes X$. $\theta^{P,X}$ is an evaluation map.

Axiom 4. (Outline.) Let $\mathcal{U}P$ be the set of upper closed subsets of P. It is a frame. It can be shown that ΩX^P is order isomorphic to the frame coproduct of ΩX and $\mathcal{U}P$. (This is done by recalling that frame coproduct is the same as suplattice tensor, see e.g. III, 2 [JoyTie 84].) Hence $\Omega X^P \cong \mathcal{U}P +_{\mathbf{Fr}} \Omega X$ from which Axiom 4 is immediate.

Axiom 5. $\mathbf{Fr}(\mathcal{U}P, \Omega 1) \cong idl(P)$ via a routine lattice theoretic calculation. Note that $\Omega 1 \equiv P\{*\}$, i..e the power set of the singleton set. ☐

Since $\mathcal{U}P$ is a frame we can define a locale, $Idl(P)$, by $\Omega Idl(P) \equiv \mathcal{U}P$. Therefore $P \otimes X = Idl(P) \times X$ for any poset P and locale X. Note that given Axiom 4 we must have that $P \otimes X = Idl(P) \times X$ for every poset P and object X where we are defining $Idl(P)$ by

$$Idl(P) \equiv P \otimes 1.$$

Axiom 4 further implies that the monotone map $\theta^{P,X} : P \to \mathbf{C}(X, Idl(P) \times X)$ is always given by

$$p \longmapsto (X \overset{(\theta^{P,1}(p)\circ !^X, 1_X)}{\to} Idl(P) \times X).$$

In other words we insist, from Axiom 4, that the isomorphism $(P \otimes X) \times Y \cong P \otimes (X \times Y)$ is the canonical one. We shall use the notation $\downarrow : P \to \mathbf{C}(1, Idl(P))$ for $\theta^{P,1}$ since this is consistent with the localic intuition.

Notice that $Idl(P)$ will inherit any lattice structure true of P; this can in fact be shown by only appealing to the description of $Idl(P)$ as a tensor commuting with product.

Lemma 3. *Given an order enriched category* $\mathbf{C}$ *with finite products and tensors commuting with products as in Axiom 4. Then* $Idl(L)$ *is an internal distributive lattice for any distributive lattice* L*.*

Proof. From the definition of tensor we have that for any posets P_1 and P_2 and any object X,

$$P_1 \otimes (P_2 \otimes X) \cong (P_1 \times P_2) \otimes X.$$

Taking $X = 1$ and $P_1 = P_2 = L$ we have that

$$Idl(L \times L) \cong Idl(L) \times Idl(L)$$

using $L \otimes Idl(L) \cong L \otimes (1 \times Idl(L)) \cong (L \otimes 1) \times Idl(L) \cong Idl(L) \times Idl(L)$. $Idl(\_)$ can then be seen to preserve finite products and so preserves the property of being a distributive lattice. □

If P is the two element poset $\mathbf{2} \equiv \{0 \leq 1\}$ then the notation $\mathbb{S}$ is used for $Idl(\mathbf{2})$. It follows that $\mathbb{S}$ is always an internal distributive lattice since $\mathbf{2}$ is a distributive lattice. Notice that, for the category of locales, $\mathbf{Loc}(X, \mathbb{S}) \cong \Omega X$ since (it can be verified that) frame homomorphisms from $\mathcal{U}\mathbf{2}$ to ΩX are in bijection with ΩX. It is then clear that Axioms 6,7 and 8 are true of $\mathbf{Loc}$.

4. Main Result

The aim of this section is to show that given $\mathbf{C}$ satisfying all the axioms, then $\mathbf{C}$ is equivalent to the category of locales.

Firstly we verify that the Axioms 1-5 are sufficient to show that $\mathbf{C}(X, \mathbb{S})$ is a frame for every object X of $\mathbf{C}$. We know already, axiomatically, that $\mathbf{C}(X, \mathbb{S})$ is a distributive lattice since it inherits this structure from $\mathbb{S}$ and, by Axiom 1, that it has directed joins. It remains to prove that finite meets distribute over them.

Lemma 4. *Assuming Axioms 1-5,* $\mathbf{C}(X, \mathbb{S})$ *is a frame for every object* X *of* $\mathbf{C}$*.*

Proof. We first give an explicit construction for the join of a directed subset $I \subseteq^{\uparrow} \mathbf{C}(X, \mathbb{S})$. Without loss of generality assume that I is lower closed. Then I is the maximal element of $idl(I)$. Axiom 5 then informs us that $\mathbf{C}(1, Idl(I))$ has a maximal element, $\top : 1 \to Idl(I)$. Let $\widehat{I} : Idl(I) \times X \to \mathbb{S}$ be the mate of the inclusion $I \to \mathbf{C}(X, \mathbb{S})$. We now verify that $\widehat{I} \circ (\top \circ !^X, 1_X) : X \to \mathbb{S}$ is the (directed) join of the subset I. Certainly $\widehat{I} \circ (\top \circ !^X, 1_X)$ is an upper bound since we know that $I \to \mathbf{C}(X, \mathbb{S})$ can be factored as $I \overset{\theta^{I,X}}{\to} \mathbf{C}(X, Idl(I) \times X) \overset{\mathbf{C}(1_X, \widehat{I})}{\to} \mathbf{C}(X, \mathbb{S})$ and it has been commented on already that $\theta^{I,X}(i) = (\downarrow i \circ !^X, 1_X)$. On the other hand given $a : 1 \to \mathbf{C}(X, \mathbb{S})$ with $i \leq a$ for every $i \in I$, then a factors as

$$X \overset{(\downarrow i \circ !^X, 1_X)}{\to} Idl(I) \times X \overset{\widehat{a \circ !^I}}{\to} \mathbb{S}.$$

for every $i \in I$. But $\top = \bigvee^{\uparrow}_{i \in I} \downarrow i$ and so, since $\mathbf{C}$ is dcpo enriched we have that a factors as

$$X \overset{(\top \circ !^X, 1_X)}{\to} Idl(I) \times X \overset{\widehat{a \circ !^I}}{\to} \mathbb{S}$$

from which $\widehat{I} \circ (\top \circ !^X, 1_X) \leq a$ since $I \to \mathbf{C}(X, \mathbb{S}) \leq a \circ !^I$ by the assumption that $i \leq a$ for every $i \in I$.

To prove that finite meets distribute over directed joins, given a directed indexing $l : I \to \mathbf{C}(X, \mathbb{S})$ and given $a : 1 \to \mathbf{C}(X, \mathbb{S})$ note that the indexing for the join $\bigvee^{\uparrow}\{a \wedge i \mid i \in I\}$ is

$$I \overset{(a \circ !^I, l)}{\to} \mathbf{C}(X, \mathbb{S}) \times \mathbf{C}(X, \mathbb{S}) \cong \mathbf{C}(X, \mathbb{S} \times \mathbb{S}) \overset{\mathbf{C}(1_X, \sqcap)}{\to} \mathbf{C}(X, \mathbb{S})$$

and so by naturality of tensor we have that the mate of this is

$$Idl(I) \times X \overset{(\widehat{a \circ !^I}, \widehat{l})}{\to} \mathbb{S} \times \mathbb{S} \overset{\sqcap}{\to} \mathbb{S}.$$

This proves distributivity by 'evaluating' (i.e. precomposing) this map at $X \overset{(\top \circ !^X, 1_X)}{\to} Idl(I) \times X$. □

Certainly given $f : X \to Y$, a morphism in $\mathbf{C}$, we have that $\mathbf{C}(f, \mathbb{S}) : \mathbf{C}(Y, \mathbb{S}) \to \mathbf{C}(X, \mathbb{S})$ is a distributive lattice homomorphism. But it is also a dcpo homomorphism by the assumption, Axiom 1, that $\mathbf{C}$ is dcpo enriched. Therefore using only Axioms 1-5 we can define a functor

$$F : \mathbf{C} \to \mathbf{Loc}$$

and Axiom 6 is exactly the assertion that this functor is full and faithful. To prove that F is an equivalence we must verify that it is essentially surjective.

Theorem 1. *Given a category $\mathbf{C}$, satisfying Axioms 1-8, we have that $\mathbf{C} \simeq \mathbf{Loc}$.*

Proof. We must verify that F is essentially surjective; that is, for every frame ΩX we must find an object $\overline{X}$ of $\mathbf{C}$ such that $\Omega X \cong \mathbf{C}(\overline{X}, \mathbb{S})$. Now Lemma 1 above shows that ΩX is the frame coequalizer of a pair of frame homomorphism

$$idl(\overline{R}) \underset{\Omega e_2}{\overset{\Omega e_1}{\rightrightarrows}} idl(L)$$

for two distributive lattices $\overline{R}$ and L. But for any distributive lattice, N, we have (using Axiom 8) that

$$idl(N) \cong \mathbf{C}(Spec(N), \mathbb{S}).$$

Using Axiom 6 we therefore have (unique) maps $\overline{e_1}, \overline{e_2} : Spec(L) \to Spec(\overline{R})$ such that $\Omega e_i \cong \mathbf{C}(e_i, \mathbb{S})$ $i = 1, 2$. Define $\overline{X}$ to be the equalizer in $\mathbf{C}$ of the diagram

$$Spec(L) \underset{e_2}{\overset{e_1}{\rightrightarrows}} Spec(\overline{R}).$$

It follows from Axiom 7 that $F(\overline{X}) \cong \Omega X$ and so we are done. □

A similar approach should be possible for the category of Grothendieck toposes. The presheaf category $[\mathbb{C}, \mathbf{Set}]$ would take the role of the tensor $\mathbb{C}\otimes 1$ and for any coherent category $\mathbb{C}$ the sheaf category $Sh(\mathbb{C}, P)$, where P is the coherent topology, would take the role of $Spec(\mathbb{C})$. This would be a subject for further work.

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