A Generalized Coverage Theorem

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Abstract

We express both Johnstone's original coverage theorem [Joh82] and its preframe version as facts about the creation of coequalizers in the category of frames. The former constructs a coequalizer from a particular coequalizer in the underlying cateogry of SUP-lattices, the latter does the same but uses the underlying category of preframes. We see that both these coverage theorems have the same form applied to two different symmetric monoidal closed categories. We state and prove a coverage result for any symmetric monoidal closed category and then look at examples. These cover all the well known coverage results but yield something new in the case when the underlying symmetric monoidal closed category is that of directed complete partial orders. The coverage theorem then tells us that preframe presentations present.

1 Introduction

Johnstone's coverage theorem [Joh82] gives us a concrete description of the frame corresponding to a set of generators and frame relations. The fact that such a frame exists can be verified easily enough by constructing the free frame on the generators and then quotienting by the least congruence containing the relations. However the advantage of the coverage theorem is that it gives us a concrete description of the frame being presented. Hence we have a concrete description of arbitary frame coproduct, and this can then be used to prove that the coproduct of compact frames is compact. In other words the product of compact locales is compact (i.e. localic Tychonoff's theorem). It was observed in the work of Abramsky and Vickers [AV93] that the real content of the coverage theorem is the fact that the frame being presented is isomorphic to the free SUP-lattice on another set of generators and relations. This ability to describe frames as particular quotients of free SUP-lattices is useful in the context of quantales since there one it often trying to find SUP-lattice homomorphisms away from a particular frame. Infact the coverage theorem extends very naturally to become a statement about how to construct coequalizers in the category of quantales.

The proof of the localic Tychonoff theorem using Johnstone's original description of the coproduct frame is far from straight forward. Many attempts were made to simplify e.g.[JV91]. In [JV91] the authors develop the theory of preframes, and check that given a set of generators and preframe relations then the preframe being presented is well defined. It is then possible to find a preframe version of the coverage theorem: it states that given a set of generators and frame relations then the frame being presented is isomorphic to the preframe being presented by some other set of generators and relations. Just as was done with the original coverage theorem this preframe version can be used to give an explicit description of the coproduct of frames. Only now the copoduct is presented as a preframe and it turns out that the proof of the localic Tychonoff theorem is much simpler.

Section 2 of this paper is dedicated to an exposition of how both Johnstone's original coverage theorem and its preframe version can be viewed as statements about the existence of coequalizers in the category **Frm** of frames. Firstly these coequalizers are seen to be isomorphic to certain coequalizers in the category **SUP** of SUP-lattices and secondly they are seen as particular coequalizers in the category **PreFrm** of preframes. The section will assume that presentations of SUP-lattices and preframes do present, and consequently that both these categories are symmetric monoidal closed. i.e. they have a tensor $\otimes$ and $A \otimes (\_)$ has a right adjoint for every object A.

We then go on to state the general coverage theorem. This is a theorem about any symmetric monoidal closed category $\mathcal{C}$ and so in the above we are thinking of $\mathcal{C}$ as **SUP** and **PreFrm** respectively. The theorem gives an explicit description of how to construct coequalizers in the cateogry of commutative monoids over $\mathcal{C}$. They are described as particular coequalizers in $\mathcal{C}$.
The statement and proof of the theorem is quite staightforward. Once it is known then not only do we find that the original coverage theorems follow, but so will the coverage theorem for quantales. We are able to see that a well known result from ring theory is just another example of the coverage theorem. Finally we can apply the theorem to the symmetric monoidal cateogry **dcpo** of directed complete partial orderes (i.e. partial orders with directed joins). The coverage theorem then shows us how to construct coequalizers in **PreFrm** and so we deduce that preframe presentations present thereby eliminating the need for any lengthy discussion about whether such presentations do actually define a preframe. Of course such a conlusion requires the well known fact that **dcpo** has coequalizers. For completeness we outline how this is checked.

2 Coverage Results

Johnstone's coverage theorem as stated in [Joh82] is as follows: given a semilattice A and a function $C : A \to PPA$ (the *coverage*) such that $C(a) \subseteq \downarrow a \quad \forall a \in A$ and C is *meet stable* (i.e. $\forall a \in A, \forall T \in C(a), \forall b \in A$ we have $\{t \wedge b | t \in T\} \in C(a \wedge b)$) then if $C - Idl(A)$ is the set of *C-ideals* of A (i.e. the lower closed subsets I of A such that $\forall a \in A, \forall T \in C(a)$ if $T \subseteq I$ then $a \in I$) then $C{-}Idl(A)$ is a frame and $A \xrightarrow{\text{<->}} C{-}Idl(A)$ (which is defined to take $a \in A$ to the ideal completion of $\{a\}$) is the free semilattice homomorphism from A to a frame which *takes covers to joins*. (i.e. if B is a frame and $f : A \to B$ then f takes covers to joins iff $\forall a \in A, \forall T \in C(a) \quad \bigvee_B \{f\bar{a} | \bar{a} \in T\} = fa$.)
When Abramsky and Vickers were investigating quantales in [AV93] they found it useful to view the coverage theorem as the statement that certain frame presentation could equally be viewed as SUP-lattice presentations. Indeed in the 'Preframe Presentation Presents' paper [JV91] the *content* of the coverage result is stated as follows: given any meet semilattice A with a coverage on it then

$$\begin{aligned} &\mathsf{Frm} < A \text{ (qua meet semilattice) } | a = \vee T \quad T \in C(a) > \\ &\cong \mathsf{SUP} < A \text{ (qua poset) } | a = \vee T \quad T \in C(a) > \end{aligned}$$

We take Johnstone's coverage theorem to be this last result and prove that it implies and is implied by the *coequalizer result*. This is defined to be the following: if

$$B \underset{g}{\overset{f}{\rightrightarrows}} A$$

is a diagram in **Frm** and if

$$B \otimes A \underset{\wedge(g \otimes 1)}{\overset{\wedge(f \otimes 1)}{\rightrightarrows}} A \xrightarrow{\quad e \quad} E$$

is a coequalizer diagram in **SUP** then

$$B \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{\;e\;} E$$

is a coequalizer diagram in **Frm**.
We now assume Johnstone's coverage theorem and try to prove the coequalizer result. Say we are given

$$B \underset{g}{\overset{f}{\rightrightarrows}} A$$

in **Frm**. Define a coverage on A as follows:

$$\begin{array}{rl} \{fb \wedge a\} \in C(gb \wedge a) & \forall b \in B, \forall a \in A \\ T \in C(\bigvee_A T) & \forall T \subseteq A \end{array}$$

(It is easy to check that this defines a coverage.)

But it is clear that with this coverage then the coequalizer of

$$B \underset{g}{\overset{f}{\rightrightarrows}} A$$

(in **Frm**) must be the frame presented by

$$\mathsf{Frm}< A \text{ (qua meetsemilattice) } | a = \vee T \quad T \in C(a) >$$

and also that the coequalizer of

$$B \otimes A \underset{\wedge(g \otimes 1)}{\overset{\wedge(f \otimes 1)}{\rightrightarrows}} A$$

(in **SUP**) must be the SUP-lattice presented by

$$\mathsf{SUP}< A \text{ (qua poset) } | a = \vee T \quad T \in C(a) >$$

so an assumption of the coverage theorem allows us to conclude the coequalizer result.

Conversely let us assume the coequalizer result. Say we are given a coverage $C : A \to PPA$ on some semilattice A. Let DA be the set of lower closed subsets of A. It is clearly a frame where join is given by union and meet is given by intersection. It is also the free frame on the semilattice A. Let B be the least congruence on $DA \times DA$ which contains $(\downarrow T, \downarrow a)$ for all pairs (T, a) such that $T \in C(a)$. So there are frame homomorphisms

$$B \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} DA$$

It is easy to see that if their coequalizer exists then it is

$$\mathsf{Frm}< A \text{ (qua meetsemilattice) } | a = \vee T \quad T \in C(a) >.$$

But

Lemma 2.1 *The free SUP-lattice on A qua poset and the free frame on A qua meet semilattice are isomorphic.*

Proof: They are both given by DA. □
Because of this fact we know that there is a SUP-lattice morhpism e from DA to the SUP-lattice E defined to be

$$\mathsf{SUP}< A \text{ (qua poset) } | a = \vee T \quad T \in C(a) >.$$

It is easy to verify that

$$B \otimes DA \underset{\wedge(\pi_2 \otimes 1)}{\overset{\wedge(\pi_1 \otimes 1)}{\rightrightarrows}} DA \xrightarrow{\ e\ } E$$

is a coequalizer diagram in **SUP** and so Johnstone's coverage theorem will follow from the coequalizer result.

Before we tackle the preframe version of the coverage theorem we need to make an observation about the free $\wedge$-semilattice on a poset. Once the material of the rest of the paper is understood it will be clear that this lemma is just another example of a result of the type: 'it is possible to lift colimits to the category of commutative monoids on a commutative monoidal closed category'. Here the commutative monoidal category is the category of meet semilattices.

Lemma 2.2 *A is a join semilattice. Then the free meet semilattice on A qua poset (i.e. $\mathsf{SLat} < A | a_1 \wedge a_2 = a_1$ if $a_1 \leq_A a_2 >$) is a distributive lattice and is the free distributive lattice on A qua $\vee$-semilattice (i.e. $\mathsf{DLat} < A | a_1 \vee a_2 = a_1 \vee_A a_2 \quad \forall a_1, a_2 \in A, \quad 0 = 0_A >$).*

Proof: Set $B = \mathsf{SLat} < A(\text{qua poset}) >$. It is well known that product and coproduct coincide in **SLat**. The coproduct of B is given by the $\wedge$-semilattice tensor. i.e.

$$B \coprod B \cong B \otimes B \equiv \mathsf{SLat} < a \otimes \bar{a} \quad (a, \bar{a}) \in A | \text{tensor equations} >$$

and product can be seen to be given by

$$B \times B \cong \mathsf{SLat} < (a, \bar{a}) \in A \times A | (a_1, \bar{a}_1) \leq (a_2, \bar{a}_2) \text{ if } a_1 \leq_A a_2, \bar{a}_1 \leq_A \bar{a}_2 >$$

Given this last universal characterization of $B \times B$ it is easy to see that

$$\begin{array}{ccc} B \times B & \xrightarrow{\ \vee\ } & B \\ \uparrow & & \uparrow \\ A \times A & \xrightarrow{\ \vee_A\ } & A \end{array}$$

defines a joins operation on B. But since $B \times B$ is also tensor we can see that the join distributes with meet in the appropriate way. Hence B is a distributive lattice. $\Box$

The preframe version of the coverage theorem (5.1 of [JV91]) is as follows: Let A be a join semilattice and let C be a set of preframe relations of the form

$$\wedge S \leq \bigvee\nolimits_i^{\uparrow} \wedge S_i$$

which are *join stable*. This means that if $x \in A$ and $\wedge S \leq \bigvee_i^{\uparrow} \wedge S_i$ is in C than

$$\wedge\{x \vee y : y \in S\} \leq \bigvee\nolimits_i^{\uparrow} \wedge\{x \vee y : y \in S_i\}$$

is also in C.
Then

$$\mathsf{PreFrm} < A \text{ (qua poset) } | C > \cong \mathsf{Frm} < A \text{ (qua } \vee\text{-semilattice)} | C >$$

the generators corresponding under the isomorphism in the obvious way.

The preframe version of the coequalizer result is as follows: if

$$B \underset{g}{\overset{f}{\rightrightarrows}} A$$

is a diagram in Frm and if

$$B \otimes A \underset{\vee(g\otimes 1)}{\overset{\vee(f\otimes 1)}{\rightrightarrows}} A \xrightarrow{\quad e \quad} E$$

is a coequalizer diagram in PreFrm then

$$B \underset{g}{\overset{f}{\rightrightarrows}} A \xrightarrow{\quad e \quad} E$$

is a coequalizer diagram in Frm.

Let us assume the preframe coverage theorem. Say we are given

$$B \underset{g}{\overset{f}{\rightrightarrows}} A$$

in Frm. Define C, a set of preframe relations on A, as follows:

$$\bigvee_A^{\uparrow} J \leq \bigvee^{\uparrow}\{j | j \in J\}$$

for every directed $J \subseteq^{\uparrow} A$ and

$$a_1 \wedge a_2 \leq a_1 \wedge_A a_2 \quad \forall a_1, a_2 \in A$$

and $\forall b \in B, \forall a \in A$

$$fb \vee a \leq gb \vee a$$
$$gb \vee a \leq fb \vee a$$

It is easy to see that C is join stable. It is also easy to see that

$$\mathsf{PreFrm} < A \text{ (qua poset) } |C >$$

is the coequalizer of

$$B \otimes A \underset{\vee(g\otimes 1)}{\overset{\vee(f\otimes 1)}{\rightrightarrows}} A$$

in PreFrm and that

$$\mathsf{Frm} < A \text{ (qua } \vee\text{-semilattice) } |C >$$

is the coequalizer of

$$B \underset{g}{\overset{f}{\rightrightarrows}} A$$

in Frm. Hence the preframe coequalizer result follows from the preframe coverage theorem.

If we look at the case of the preframe coverage theorem when C is the empty set, it is then the statement that the free preframe on a poset A is equal to the free frame on the join semilattice A if A is indeed a join semilattice. But such a free preframe can be seen to be the ideal completion of the free semilattice on the poset A, and such a free frame can be seen to be the ideal completion of the free distributive lattice on the join semilattice A. But since lemma 2.1 showed us that the free semilattice and the free distributive lattice just described are the same we know that their ideal completions are isomorphic. Hence we have proven the preframe coverage theorem in the case when C is empty. i.e.

Lemma 2.3 *The free preframe on a poset A is isomorphic to the free frame on A qua join semilattice provided A is indeed a joins semilattice.* □

Given a join semilattice A we will call the free frame on it K_A. The fact that it is also a free preframe will help us prove that the preframe coequalizer result implies the coverage theorem.

Say we are given a join semilattice A and a join stable collection of preframe relations C. Let $j : A \rightarrowtail K_A$ denote the inclusion of generators. Let B be the least congruence on K_A which contains all the pairs

$$(\wedge_{K_A}\{js : s \in S\}, (\wedge_{K_A}\{js : s \in S\}) \wedge_{K_A} (\bigvee_i^{\uparrow} \wedge_{K_A}\{js | s \in S_i\}))$$

So there are two frame inclusions

$$B \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} K_A$$

and it is easy to see that there coequalizer in Frm is $\mathsf{Frm}< A$ (qua $\vee$-semilattice)$|C >$. Further more once we view K_A as the free preframe on A (qua poset) then it can be seen that the coequalizer of

$$B \otimes K_A \underset{\vee(\pi_2 \otimes 1)}{\overset{\vee(\pi_1 \otimes 1)}{\rightrightarrows}} K_A$$

is equal to $\mathsf{PreFrm}< A$ (qua poset) $|C >$. Hence the coverage theorem follows from the coequalizer result.

Of course it a matter of opinion as to whether the coequalizer results really capture the coverage results, particularly in view of the need for lemmas 2.1 and 2.2.

3 General Coverage Theorem

Given a symmetric monoidal category $\mathcal{C}$ with tensor $\otimes$ it is well known that the category $\mathsf{CMon}(\mathcal{C})$ of commutative monoids over $\mathcal{C}$ has finite coproducts. Its initial object will just be the unit of the tensor (written Ω), and if $(A *_A, e_A), (B, *_B, e_B)$ are two objects of $\mathsf{CMon}(\mathcal{C})$ then their coproduct will be $(A \otimes B, *_{A\otimes B} e_{A\otimes B})$ where $*_{A\otimes B}$ is given by the composite

$$(A \otimes B) \otimes (A \otimes B) \xrightarrow{\cong} (A \otimes A) \otimes (B \otimes B) \xrightarrow{*_A \otimes *_B} A \otimes B$$

and $e_{A\otimes B}$ is the composite

$$\Omega \xrightarrow{\cong} \Omega \otimes \Omega \xrightarrow{e_A \otimes e_B} A \otimes B$$

To see that this defines coproduct note that with these assignments we can view $\otimes$ as a functor from $\mathsf{CMon}(\mathcal{C}) \times \mathsf{CMon}(\mathcal{C})$ to $\mathsf{CMon}(\mathcal{C})$. It can be seen to be left adjoint to the diagonal functor

$$\Delta : \mathsf{CMon}(\mathcal{C}) \longrightarrow \mathsf{CMon}(\mathcal{C}) \times \mathsf{CMon}(\mathcal{C})$$

by checking the triangle identities and noting that

$$A \xrightarrow{\cong} A \otimes \Omega \xrightarrow{1 \otimes e_B} A \otimes B$$

$$B \xrightarrow{\cong} \Omega \otimes B \xrightarrow{e_A \otimes 1} A \otimes B$$

defines the unit and $*_A : A \otimes A \to A$ defines the counit.

If we assume further that $\mathcal{C}$ is symmetric monoidal *closed* (i.e. that for every object A of $\mathcal{C}$ $A \otimes (\_)$ has a right adjoint) then not only does $\mathsf{CMon}(\mathcal{C})$ have all finite coproducts but it also has all filtered colimits provided $\mathcal{C}$ does:

the forgetful functor $F : \mathsf{CMon}(\mathcal{C}) \to \mathcal{C}$ creates filtered colimits. The proof is quite straight forward. Say $D : J \longrightarrow \mathsf{CMon}(\mathcal{C})$ is a filtered diagram in $\mathsf{CMon}(\mathcal{C})$. Since $\otimes$ preserves colimits in each of its coordinates we can do the following manipulations:

$$\begin{aligned} colim_i FD(i) \otimes colim_j FD(j) &\cong colim_i (FD(i) \otimes colim_j (FD(j)) \\ &\cong colim_i (colim_j (FD(i) \otimes FD(j))) \\ &\cong colim_{(i,j)} FD(i) \otimes FD(j) \end{aligned}$$

But from a piece of well known ‘abstract nonsense’ [Mac71] we know that

$$colim_{(i,j)}(D(i) \otimes D(j)) \cong colim_i (FD(i) \otimes FD(i))$$

since J is a filtered category and so the monoid operation $*_{D(i)}$ on the $D(i)$s induce a function

$$*_D : colim_i FD(i) \otimes colim_i FD(i) \to colim_i FD(i)$$

As for a unit on $colim_i FD(i)$ note that the composite

$$\Omega \xrightarrow{e_{D(i)}} FD(i) \xrightarrow{\coprod_{FD(i)}} colim_i FD(i)$$

(where the $\coprod_{FD(i)}$ is an edge of the colimit cocone on FD) are the same for every i (use filtered-ness of J) and so define a unit (e_D) for $colim_i FD(i)$. It is then easy to check that $(colim_i FD(i), *_D, e_D)$ is the colimit of D in $\mathsf{CMon}(\mathcal{C})$. So to complete our discussion about the existence of colimits in the category $\mathsf{CMon}(\mathcal{C})$ all we need to do is find out whether coequalizers exists or not. It turns out that the answer to this question, which is contained in the next theorem, can also be understood as a generalized coverage theorem.

Theorem 3.1 *If $\mathcal{C}$ is a symmetric monoidal closed category and*

$$(A, *_A, e_A) \underset{g}{\overset{f}{\rightrightarrows}} (B, *_B, e_B)$$

is a diagram in CMon($\mathcal{C}$) then if $c : B \to C$ is the coequalizer of

$$A \otimes B \underset{*_B(g \otimes 1)}{\overset{*_B(f \otimes 1)}{\rightrightarrows}} B$$

then C can be given a commutative monoid structure $(C, *_C, e_C)$ such that

$$(A, *_A, e_A) \underset{g}{\overset{f}{\rightrightarrows}} (B, *_B, e_B) \xrightarrow{c} (C, *_C, e_C)$$

is a coequalizer diagram in CMon($\mathcal{C}$).

Proof: The definition of e_C is just the composite $c \circ e_B$. Defining $*_C$ is a little more involved. Since $\mathcal{C}$ is closed we know that the endofunctor $(\_) \otimes B$ preserves coequalizers, hence the diagram

$$A \otimes B \otimes B \underset{(* \otimes 1)(g \otimes 1 \otimes 1)}{\overset{(* \otimes 1)(f \otimes 1 \otimes 1)}{\rightrightarrows}} B \otimes B \xrightarrow{c \otimes 1} C \otimes B$$

(where $* = *_B$) is a coequalizer diagram in $\mathcal{C}$. But by associativety of the commutative monoid B the morphisms $(* \otimes 1) \circ (f \otimes 1 \otimes 1)$ and $(* \otimes 1) \circ (g \otimes 1 \otimes 1)$ are equalized by the morphism

$$B \otimes B \xrightarrow{*} B \xrightarrow{c} C$$

and so there exists a (unique) map $d : C \otimes B \to C$ such that $d \circ (c \otimes 1) = c \circ *$. So if $\tau : B \otimes C \to C \otimes B$ is the flip isomporhism certainly the composite $d \circ \tau$ coequalizes $(1 \otimes c) \circ (* \otimes 1) \circ (f \otimes 1 \otimes 1)$ and $(1 \otimes c) \circ (* \otimes 1) \circ (g \otimes 1 \otimes 1)$. But we have 2 commutative squares:

$$\begin{array}{ccc} A \otimes B \otimes B & \underset{(* \otimes 1)(g \otimes 1 \otimes 1)}{\overset{(* \otimes 1)(f \otimes 1 \otimes 1)}{\rightrightarrows}} & B \otimes B \\ {\scriptstyle 1 \otimes 1 \otimes c} \downarrow & & \downarrow {\scriptstyle 1 \otimes c} \\ A \otimes B \otimes C & \underset{(* \otimes 1)(g \otimes 1 \otimes 1)}{\overset{(* \otimes 1)(f \otimes 1 \otimes 1)}{\rightrightarrows}} & B \otimes C \end{array}$$

and so since $1 \otimes 1 \otimes c$ is an epimorphism (as c is) we know that $d \circ \tau$ will factor throught the coequalizer of the bottom row. But the coequalizer of the bottom row is $c \otimes 1 : B \otimes C \to C \otimes C$ since $(\_) \otimes C$ preserves coequalizers. Hence there exists $*_C : C \otimes C \to C$ such that $d \circ \tau = *_C \circ (c \otimes 1)$. It is now a routine exercise to check that $(C, *_C, e_C)$ is a commutative monoid, that c is a commutative monoid homomorphism and that

$$(A, *_A, e_A) \underset{g}{\overset{f}{\rightrightarrows}} (B, *_B, e_B) \xrightarrow{c} (C, *_C, e_C)$$

is a coequalizer diagram in $\mathsf{CMon}(\mathcal{C})$ as required. For instance since $d\tau = *_C(c \otimes 1)$ we have that $d = *_C(1 \otimes c)$ and so $*_C(c \otimes c) = *_C(1 \otimes c)(c \otimes 1) = d(c \otimes 1) = c*$. i.e. c is a monoid homorphism. Also $(c \otimes c \otimes c)$ is epic and so associativety for $*_C$ follows from associativety of $*$.

□

There is a forgetful functor going from $\mathsf{CMon}(\mathcal{C})$ to $\mathcal{C}$. This functor has a left adjoint if and only if free commutative monoids can be found on $\mathcal{C}$ objects. We find, converse to the coverage theorem, that if there is some category $\mathcal{D}$ and a faithful functor U from $\mathcal{D}$ to $\mathcal{C}$ with a left adjoint then coequalizers in $\mathcal{C}$ can be constructed from particular coequalizers in $\mathcal{D}$. Specifically we need to know that $\mathcal{C}$ has pullbacks and image factorizations. Frame presentations presentations present and it is easy to construct pullbacks and image factorizations in the category PreFrm of preframes (for the latter just take the subpreframe generated by the set theoretic image of the function to be factorized) so this theorem will prove that PreFrm has coequalizers. Indeed the proof to follow is really just a repitition of the proof in [JV91] that the category of preframes has coequalizers.

Theorem 3.2 *If $\mathcal{C}$ has limits and image factorizations, and there is some category $\mathcal{D}$ with a functor $U : \mathcal{D} \to \mathcal{C}$ which has a left adjoint F (with monic unit) then for any diagram*

$$B \underset{g}{\overset{f}{\rightrightarrows}} A$$

in $\mathcal{C}$ its coequalizer is given by the image factorization of $B \overset{\eta_A}{\rightarrowtail} UFB \overset{Ue}{\to} UE$ where $FB \overset{e}{\to} E$ is the coequalizer in $\mathcal{D}$ of

$$FA \underset{Fg}{\overset{Ff}{\rightrightarrows}} FB$$

Proof: Let the image factorization described in the statment be written $q : B \to ei[B]$. Say there is a morphism $B \overset{\bar{e}}{\to} \bar{E}$ in $\mathcal{C}$ such that $\bar{e}f = \bar{e}g$. So certainly $F\bar{e}Ff = F\bar{e}Fg$ and so there is a morphism d of $\mathcal{D}$

$$d : E \longrightarrow F\bar{E}$$

such that $de = F\bar{e}$. Pull the monomorphism $\eta_{\bar{E}}$ back along Ud to find a monomorphism $i : J \rightarrowtail UE$. But from the pullback diagram we see that the map $B \overset{\eta_B}{\rightarrowtail} UFB \overset{Ue}{\to} UE$ factors through i and hence the subobject J contains the subobject $ei[B]$ and so there is a map $\bar{d}$ from $ei[B]$ to $\bar{E}$ such that $\bar{d}q = \bar{e}$. Uniqueness of $\bar{d}$ follows if q is an epimorphism; but we have equalizers in $\mathcal{C}$ and so the cover q is an epimorphism. □

4 Examples

The fact that SUP has coqualizers is shown in [JT84]. In proposition 4.3 of chapter 1 they show that if R is any subset of $M \times M$ where M is a SUP-lattice then the quotient of M by the congruence generated by R is given by the set

$$Q = \{x \in M | \forall (z_1, z_2) \in R, \quad z_1 \leq x \quad \Leftrightarrow \quad z_2 \leq x\}$$

So if

$$B \underset{g}{\overset{f}{\rightrightarrows}} A$$

is a pair of arrows in SUP then use the relation $\{(fb, gb) | b \in B\}$ to define the coequalizer of f and g. Clearly we can also use this general construct to describe tensor product of SUP-lattices and so we see that SUP is a symmetric monoidal closed category with coequalizers.

A quantale is a SUP-lattice A together with a commutative monoidal structure

$$e \in A$$
$$* : A \times A \longrightarrow A$$

with the property that $*$ preserves arbitary joins in both of its coordinates. In other words a quantale is an object of CMon$(\mathcal{C})$ where $\mathcal{C}$ is the symmetric monoidal closed category of SUP-lattices. A good reference for quantales is [Ros90]. They are investigated in [AV93] as models for various process calculi. In that investigation a coverage theorem for quantales is developed. Given a commutative monoid A we say that $C : A \to PPA$ is a coverage if and only if $\forall T \in C(a), \forall b \in A$

$$\{t *_A b | t \in T\} \in C(a *_A b).$$

The coverage theorem for quantales is then the statement that the presentation

$$\text{Qu}< S \text{ (qua monoid) } | \vee T \geq a \quad \forall T \in C(a) >$$

is well defined and is isomorphic as a poset to

$$\text{SUP}< S | \vee T \geq a \quad \forall T \in C(a) >$$

The free SUP-lattice on a set S is the power set of S. But:

Lemma 4.1 *The free quantale on a monoid S (i.e. Qu$< S$ (qua monoid) $>$) is isomorphic as a poset to the free SUP-lattice on the set S.*

Proof: Both are given by PS where the operation on PS is given by, (for $T_1, T_2 \subseteq S$)

$$T_1 * T_2 = \{t_1 * t_2 | t_1 \in T_1 \quad t_2 \in T_2\} \qquad \square$$

We now proceed, exactly as in section 2, to prove that the quantale coverage result is implied by the coequalizer result. It should be clear that the coequalizer result in this case is exactly the generalized coverage theorem applied to the category $\mathcal{C}$=SUP.

Given a coverage C on some commutative monoid S let B be the least quantale congruence on PS which contains the pair

$$(T, T \cup \{a\})$$

for every $T \in C(a)$.
We then have a pair of quantale maps

$$B \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} PS$$

and it is clear that their coequalizer in Qu will be

$$\mathsf{Qu}< S \text{ (qua monoid) } | \vee T \geq a \quad T \in C(a) >$$

It is also clear that

$$\mathsf{SUP}< S | \vee T \geq a \quad T \in C(a) >$$

is the coequalizer of

$$B \otimes PS \underset{*(\pi_2 \otimes 1)}{\overset{*(\pi_1 \otimes 1)}{\rightrightarrows}} PS$$

in SUP and so the generalized coverage theorem implies the quantale coverage result.

Now say we are given a commutative monoid $(A, *, e_A)$ over a SUP-lattice $(A, \leq_A)$ which is also a semilattice. i.e. $*$ is idempotent. We can then give A a second order, with which the $*$ operation becomes meet. This second order will not necessarily coincide with $\leq_A$. However the two orders will coincide if (and only if) $a \leq_A e_A$ for every $a \in A$. For if we assume $a \leq_A e_A$ for every $a \in A$ then since $*$ is monotone in both its coordinates we know $*_A(a \otimes b) \leq_A a, b$ for every a, b. Further if $c \leq_A a, b$ then $c = *(c \otimes c) \leq_A *(a \otimes b)$ and so $*$ is meet with respect to the order $\leq_A$. Clearly such a commutative monoid will be a frame. Furthermore the reader can now return to section 2 and check that the class of commutative monoids over SUP which are idempotent and which are such that the meet-semilattice order induced is the same as the original order is closed under all the colimit constructions given. We give the details of the coequalizers case, assuming the same notations: certainly $x \leq_C e_C$ for every $x \in C$ since c is a surjection and $b \leq_B e_B$ for every $b \in B$. But since $*_C \circ (c \otimes c) = c *_B$ we see that $*_C$ is idempotent by surjectivety of c and idempotence of $*_B$. Johnstone's original coverage theorem now follows.

It would be nice to have a general result about closedness of colimits in the category of idempotent commutative monoids over some symmetric monoidal closed category. Total generality fails: if we look at the case where $\mathcal{C}$ is Abelian groups, then it is not the case that the inital object of the category of idempotent commutative monoids over Ab (= the category of Boolean algebras) is equal to the unit of Abelian group tensor. The former is the subobject classifier, the latter is the ring of integers. However the coproduct of two Boolean algebras is isomorphic to the tensor on the underlying Abelian groups but the reasoning behind this seems to follow a very different route to the reasoning contained in a proof that the coproduct of two frames is equal to their SUP-lattice tensor.

Now that it is clear how to view semilattices on objects of $\mathcal{C}$ as objects of $\mathsf{CMon}(\mathcal{C})$ for certain $\mathcal{C}$ we can put the lemmas 2.1, 2.2 and 4.1 in a general context. The need for these lemmas raised doubts as to whether we had really captured the coverage result. However they all follow a common form, and it is easy to see that there is a general result which captures them. This general result is just the statement that provided we only allow tensor preserving functors between symmetric monoidal categories then the operation $\mathsf{CMon}(\_)$ is functorial and the forgetful functor from $\mathsf{CMon}(\mathcal{C})$ to $\mathcal{C}$ for any $\mathcal{C}$ form the morphisms of a natural transformation. The commuting squares of this natural transformation give the lemmas. The categories to consider are POS (posets), SUP and Slat and the functors are 'free SUP-lattice qua poset', 'free semilattice qua poset' and 'free SUP-lattice (qua set)'.

We now turn our attention to an application of the converse of the coverage theorem (Theorem 3.2). We take $\mathcal{C}$=dcpo, the category of directed complete partial orders. It clearly has limits and image factorizations. The category $\mathcal{D}$ is taken to be SUP-lattices which we know have coequalizers. Also it is easy to see that the forgetful functor from SUP to dcpo has a left adjoint F. Simply take

$$FA = \mathsf{SUP} < A \text{ (qua dcpo) } >$$

It follows at once that dcpo has coequalizers.
From this it is easy to see how to construct dcpo tensor, and so the next step is to investigate CMon(dcpo). We know that this category has coequalizers, although it is when we restrict our attention to the idempotent commutative monoids that we get more interesting results. Provided we insist that the unit of the idempotent commutative monoid is the greatest element with respect to the original order on our dcpo A then just as in the SUP-lattice case we can see that the monoidal operation will be meet; furthermore it is a meet which commutes with directed joins in both coordinates. i.e. A has finite meets and these meets distribute directed joins; we have a preframe.
Just as in the SUP-lattice case we can now check that the colimits of these preframes are found by suitable dcpo constructions. When it comes to checking that the coequalizer of idempotent monoids is idempotent we have to take a little more care, since the coequalizing morphism is no longer necessarily a surjection. However it is easy to see that the set of idempotent elements of the coequalizer forms a subdcpo which contains the appropriate set theoretic image.

In short, preframes have coequalizers and so a preframe tensor can be defined. The next step is to look at CMon(PreFrm). We know that this category will have coequalizers, and indeed one can write a coverge theorem for it. Aside from these facts not much is known about this category as far as the author is aware. It might be possible to use it in much the same way that quantales were used as models for various process calculi in [AV93].

We restrict to the category of idempotent commutative preframe monoids. If we require the unit to be the greatest element, then we are not adding any new structure since meet will already exist. However we can look at the case when the unit is the least element with respect to the original order. This occurs exactly when the monoid operation corresponds to binary joins and so we have described frames. It follows, by a routine that should now be familiar, that the coverage theorem 3.1 in this case gives a way of constructing coequalizers in Frm, and really is just a restatement of the preframe version of the original coverage result.

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