A Universal Characterization of the Double Powerlocale

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Abstract

The double powerlocale $\mathbb{P}(X)$ (found by composing, in either order, the upper and lower powerlocale constructions P_U and P_L) is shown to be isomorphic in $[\mathbf{Loc}^{op}, \mathbf{Set}]$ to the double exponential $\mathbb{S}^{\mathbb{S}^X}$ where $\mathbb{S}$ is the Sierpiński locale. Further $P_U(X)$ and $P_L(X)$ are shown to be the subobjects $\mathbb{P}(X)$ comprising of, respectively, the meet semilattice and join semilattice homomorphisms. A key lemma shows that, for any locales X and Y, natural transformations from $\mathbb{S}^X$ (the presheaf $\mathbf{Loc}(- \times X, \mathbb{S})$) to $\mathbb{S}^Y$ (i.e. $\mathbf{Loc}(- \times Y, \mathbb{S})$) are equivalent to dcpo morphisms from the frame ΩX to ΩY. It is also shown that $\mathbb{S}^X$ has a localic reflection in $[\mathbf{Loc}^{op}, \mathbf{Set}]$ whose frame is $\mathbf{dcpo}(\Omega X, \Omega)$.

The reasoning is constructive in the sense of topos validity.

1 Introduction

1.1 Background comment on powerlocales

The convex (Plotkin), lower (Hoare) and upper (Smyth) powerdomains are well established constructions in domain theory, providing tools for the semantics of programming languages [Plotkin 83]. The convex powerdomain [Plotkin 76] is in effect an adaptation of the topological theory of hyperspaces (see [Nadler 1978]),

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but was found to embed in two more primitive powerdomains, the upper and lower [Smyth 78]. These two are less familiar in general topology, perhaps because their topologies are almost never Hausdorff. See [Schalk 93] for a summary.

All three constructions work well in localic form, giving *powerlocales* (Vietoris V [Johnstone 85], and lower P_L and upper P_U [Robinson 86]). They have been studied in particular in [Vickers 97].

It has long been known that the upper and lower powerdomain constructions commute [FlanMart 90], and in [JoVic 91] this was also proved for the upper and lower powerlocales. Their composite is what we are calling the *double powerlocale* $\mathbb{P}$. Its investigation was advocated in Section 5 of [Vickers 93], partly with a view to unifying the study of the upper and lower powerlocales. In [Vickers 95] a number of abstract results for order-enriched categories were proved, and it was shown how when these were interpreted twice in the category **Loc** of locales, once with the specialization order enrichment and once with its opposite, they yielded parallel results involving each powerlocale. However, there was nothing there to show interaction between the two powerlocales, whereas the double powerlocale encompasses both together.

Serious study of the double powerlocale started in [Vickers 02]. A major result there was that if the locale X is locally compact (hence exponentiable, see [Hyland 81] or [Johnstone 82]), then $\mathbb{P}X$ is homeomorphic to $\mathbb{S}^{\mathbb{S}^X}$, where $\mathbb{S}$ is the Sierpiński locale (i.e. the locale whose frame of opens is the free frame on the singleton set). This shows that $\mathbb{P}$, restricted to locally compact locales, is the same as the monad Σ^2 (where $\Sigma X = \mathbb{S}^X$) used extensively in Taylor's Abstract Stone Duality (see e.g. [Taylor 02]).

1.2 Objective

The objective of this paper is to prove that $\mathbb{P}X$ is homeomorphic to $\mathbb{S}^{\mathbb{S}^X}$ *even when X is not locally compact.* In other words we cover cases where $\mathbb{S}^X$ does not necessarily exist as a locale. We do this by using the Yoneda embedding of **Loc** into $[\mathbf{Loc}^{op}, \mathbf{Set}]$. We have to be careful, because **Loc** is large; in particular we do not assume that $[\mathbf{Loc}^{op}, \mathbf{Set}]$ is a topos, or even cartesian closed. However, Yoneda's Lema still holds good and we can use it to find exponentials of representable presheaves. The Yoneda embedding represents each locale X as the presheaf $\mathbf{Loc}(\_, X)$, and the Yoneda lemma tells us that Y^X is $\mathbf{Loc}(\_ \times X, Y)$. The main result here is therefore that $\mathbb{S}^{\mathbb{S}^X}$ exists in $[\mathbf{Loc}^{op}, \mathbf{Set}]$ and is given by the representable functor $\mathbf{Loc}(\_, \mathbb{P}X)$. We have thus found a characterization of the double power locale that is entirely localic. It is presentation independent, unlike the specific constructions given in [Vickers 02] by which a presentation for $\mathbb{P}X$ is constructed out of each presentation of X. On the other hand, it is also independent of the underlying lattice theory of frames, and – modulo foundational questions raised by $[\mathbf{Loc}^{op}, \mathbf{Set}]$ itself – this may have some virtue in the context of constructivist doctrines (such as predicative type theory, and the "arithmetic" logic conjectured in the conclusions of [Vickers 99]) in which frames are not admissible as sets.

We also show that the main result restricts to results about the upper and lower powerlocales. $\mathbb{S}$ has internal distributive lattice structure in **Loc** and therefore $\mathbb{S}^X$ is an internal distributive lattice in $[\mathbf{Loc}^{op}, \mathbf{Set}]$. It is shown here that the powerlocales $P_U X$ and $P_L X$ can then be identified with the subobjects of $\mathbb{S}^{\mathbb{S}^X}$ whose generalized points are (respectively) the meet and join semilattice homomorphisms from $\mathbb{S}^X$ to $\mathbb{S}$.

1.3 Proof outline

If the double exponentiation $\mathbb{S}^{\mathbb{S}^X}$ exists as a presheaf then $\mathbb{S}^{\mathbb{S}^X}(W)$ is defined as the class of natural transformations from $\mathbf{Loc}(\_, W) \times \mathbf{Loc}(\_ \times X, \mathbb{S})$ to $\mathbf{Loc}(\_, \mathbb{S})$, and these are equivalent to natural transformations from $\mathbf{Loc}(\_\times X, \mathbb{S})$ to $\mathbf{Loc}(\_\times W, \mathbb{S})$. The key technical result shown here is that these are equivalent to the dcpo morphisms between the corresponding frames of opens, i.e. from ΩX to ΩW.

Given this technical observation the main result is relatively straightforward. Recall that the defining universal frame-theoretic characterization of $\mathbb{P}X$ is that its frame is free over ΩX qua dcpo. In other words, there exists $\boxtimes : \Omega X \to \Omega \mathbb{P} X$, a universal dcpo morphism to a frame. Any dcpo morphism $q : \Omega X \to \Omega W$ extends uniquely to a frame homomorphism from $\Omega\mathbb{P}X$:

$$\begin{array}{ccc} \Omega X & \stackrel{\boxtimes}{\to} & \Omega\mathbb{P}X \\ & \searrow^{q} & \downarrow \exists!\ \Omega f \\ & & \Omega W \end{array}$$

The correspondence between natural transformations (in $[\mathbf{Loc}^{op}, \mathbf{Set}]$) and dcpo morphisms therefore allows this defining universal characterization to translate to:

$$\begin{array}{ccc} \mathbf{Loc}(\_\times X, \mathbb{S}) & \stackrel{\boxtimes}{\to} & \mathbf{Loc}(\_\times \mathbb{P}X, \mathbb{S}) \\ & \searrow^{q} & \downarrow \exists!\ \mathbf{Loc}(\_\times f, \mathbb{S}) \\ & & \mathbf{Loc}(\_\times W, \mathbb{S}) \end{array}$$

Hence the generalized points (at stage W) of $\mathbb{P}X$ are exactly the morphisms $\mathbb{S}^X \to \mathbb{S}^W$ in $[\mathbf{Loc}^{op}, \mathbf{Set}]$, i.e. exactly the maps $W \times \mathbb{S}^X \to \mathbb{S}$, i.e. the points of the double exponential of X at stage W. This proves the main result from the technical observation.

To prove that the natural transformations in question are exactly the dcpo morphisms, some basic observations about dcpo presentations are made. Specifically, a "double coverage" result for dcpos is given, allowing the reduction of frame presentations to dcpo presentations. This combines existing results whereby frame presentations are reduced to presentations of suplattices [AbrVic 93] or preframes [JoVic 91]. Suppose L^X, R^X are the generators and relations of a frame presentation for the locale X and moreover (as can always be assumed)

L^X is a distributive lattice and R^X satisfies certain "meet and join stability" conditions. Then it is shown by the double coverage result that the data for a dcpo morphism $\Omega X \to \Omega$ is a locale map

$$p : Idl(L^X) \to \mathbb{S}$$

composing equally with the maps $R^X \rightrightarrows Idl(L^X)$ given by the presentation. ($Idl(L^X)$ is the locale whose points are the ideals of L^X.) Carried out in sheaves over W (i.e. pulling back to $\mathbf{Loc}/W$) provides a description of any dcpo morphism $\Omega X \to \Omega W$, in terms of a map $Idl(L^X) \times W \to \mathbb{S}$. Finally, we make the new observations that for any locale X, $Idl(L^X)$ is a weak exponential $\mathbb{S}^X$. That is, it is an exponential without the uniqueness requirement on the transpose, see [CarbRos 00]. It then becomes routine to check that the locale map $Idl(L^X) \times W \to \mathbb{S}$ is enough data to define a natural transformation.

1.4 Notation

For notation our references are [Johnstone 82] and [Vickers 89]. For the standard "qua" notation, which generally indicates an implied application of a forgetful functor, consult, e.g. [JoVic 91]. If "qua" is used in a presentation it means "add in the equations true of the algebraic structure, so "qua $\wedge$-SemiLat" means include all the meet semilattice equations. We use **Fr, Sup, dcpo, PreFr**, **DL**, $\vee$-**SemiLat**, $\wedge$-**SemiLat** and **Pos** for the categories of frames, suplattices (complete lattices; morphisms preserve all joins), directed complete partial orders (morphisms preserve directed joins), preframes (dcpos with finite meets distributing over directed joins; morphisms preserve finite meets and directed joins), distributive lattices, join semilattices, meet semilattices and posets respectively.

2 Frames via Dcpo Presentations

The idea of presentation by generators and relations is well known from universal algebra in the case of finitary algebraic theories. It does not automatically apply to frames, because of the unbounded arities of the join operators. However, the existence of free frames makes it work ([Johnstone 82]; and for a more detailed description see [Vickers 89]).

In Johnstone's original construction of a frame from a presentation (the *coverage theorem* of Section 2.11, Ch. II in [Johnstone 82]) he requires the presentation to be in a particular form: the generators form a meet semilattice G, and the relations R use only joins and are meet stable. Then the construction has the universal property of $\mathbf{Fr}\langle G \text{ (qua } \wedge\text{-SemiLat)} | \, R\rangle$. In the original papers the relations (expressed by a coverage) were always of the form $a \leq \bigvee U$, but it is not hard to see that the discussion still holds with relations of the general form $\bigvee U = \bigvee V$. Then the meet stability requirement is that given such a presenting relation, and a generator b, then $\bigvee\{u \wedge b | u \in U\} = \bigvee\{v \wedge b | v \in V\}$ is also one of the presenting relations. These restrictions on the presentation are not significant as any presentation can be manipulated into this form (details omitted; but see [Vickers 02] for an extensive discussion).

In [AbrVic 93] it is observed that Johnstone's construction (via C-ideals) is in fact a construction of $\mathbf{Sup}\langle G \text{ (qua poset)} \mid R\rangle$ and so there it is suggested that the essence of the coverage theorem is that

$$\mathbf{Fr}\langle G \text{ (qua } \wedge\text{-SemiLat)} \mid R\rangle \cong \mathbf{Sup}\langle G \text{ (qua poset)} \mid R\rangle. \qquad \text{(CovThm)}$$

Hence the theorem can be used to transform frame presentations into suplattice presentations. For our purposes it is useful to note that once one knows that the right-hand side exists, it is possible to use its universal property to define its frame structure (here, the finite meets distributing over joins) and to prove it has the universal property of the left-hand side.

Dually (replacing finite meets with finite joins) the analogous "preframe coverage theorem" in [JoVic 91] revolves around join semilattice structure and transforms join stable frame presentations into preframe presentations. (A *preframe* has finite meets and directed joins, with distributivity of the former over the latter.) It states that for any join stable set of relations R on a join semilattice G of generators,

$$\mathbf{Fr}\langle G \text{ (qua } \vee\text{-SemiLat)} \mid R\rangle \cong \mathbf{PreFr}\langle G \text{ (qua poset)} \mid R\rangle.$$

The aim of this section is to prove a double coverage result that combines both the preframe coverage and the original suplattice coverage result. The presentation must be of the form of join and meet stable relations on a distributive lattice. Some applications of the double coverage result are given showing an explicit description of locale product in terms of a dcpo construction. This description is necessary to prove the main result.

2.1 The Double Coverage Theorem

We shall define the notion of *DL-site* which is a type of presentation for a frame. In a DL-site the generators form a distributive lattice (DL) and the relations, involving only directed joins, must have both meet and join stability. The double coverage result is that

$$\mathbf{Fr}\langle G \text{ (qua DL)} \mid R\rangle \cong \mathbf{dcpo}\langle G \text{ (qua poset)} \mid R\rangle.$$

To express the meet and join stability properties succinctly we use the idea of an L-set for any distributive lattice L. This is simply a set with two actions by L, for the monoids $(L, 0, \vee)$ and $(L, 1, \wedge)$.

Example 1 *The set $idl(L)$ of all ideals (lower closed directed subsets) of L is an L-set with actions*

$$(l, I) \longmapsto \{l \wedge m \mid m \in I\}$$
$$(l, I) \longmapsto \downarrow \{l \vee m \mid m \in I\}$$

Definition 2 1. *A* DL-site *comprises a distributive lattice L, an L-set R and a pair of L-set homomorphisms $e_1, e_2 : R \rightrightarrows idl(L)$.*

2. *A* dcpo presentation *comprises a poset P and a set R together with a pair of functions $e_1, e_2 : R \rightrightarrows idl(P)$*

Given a DL-site (L, R, e_1, e_2) then we write

$$\mathbf{Fr}\langle L \text{ (qua DL)} \mid R\rangle$$

as abbreviation for

$$\mathbf{Fr}\langle L \text{ (qua DL)} \mid \bigvee^{\uparrow} e_1(r) = \bigvee^{\uparrow} e_2(r) \ (r \in R)\rangle$$

and similarly for a dcpo presentation (P, R, e_1, e_2).

Example 3 *Any frame has a presentation by a DL-site. Given a frame ΩX, take $L^X = \Omega X$ and $R^X = idl(\Omega X)$. The L^X-set morphisms from R^X to $idl(L^X)$ are the identity and $\downarrow \circ \bigvee^{\uparrow}$. Such a presentation is referred to as the* standard presentation *for the frame. More generally,* any *frame presentation can be manipulated into a DL-site presenting the same frame (see [Vickers 02], though we shall not need the details of this in what follows).*

By definition every DL-site can also be used as a dcpo presentation; the double coverage result is that they present the same poset. To prove this it must first be checked that dcpo presentations present. That this is so seems to be folklore, though we have not found a good reference in the literature. It uses the fact that coequalizers of dcpos exist, and this has probably been known at least since [Markowsky 77]. We give a proof that reapplies the techniques of [JoVic 91], where preframe presentations are proved to present from the fact that frame presentations do. Here, we replace preframes by dcpos and frames by suplattices.

Lemma 4 *If A is a dcpo, then the free suplattice over it is provided by the set of Scott closed subsets. The injection of generators is monic.*

Proof. By *Scott closed subset* of A we understand a subset that is lower closed and closed under directed joins (in A). Let us write $F(A)$ for the set of Scott closed subsets of A. Any intersection of Scott closed subsets is clearly Scott closed and so $F(A)$ is a complete lattice. Note that the joins are not unions, but the Scott closures of unions. (Constructively, not even finitary joins of Scott closed subsets are Scott closed.) $\downarrow: A \to F(A)$ preserves directed joins and is monic, and this map will prove to be the injection of generators – the unit of the free suplattice monad on **dcpo**. To see this first note that for any $B \in F(A)$, $B = \bigvee\{\downarrow b \mid b \in B\}$ since the join always contains the set-theoretic union. So, given any dcpo morphism $\phi : A \to M$ with M a suplattice, the assignment $q(B) = \bigvee_M\{\phi(b) \mid b \in B\}$ is therefore necessary if ϕ is to factor as $q\circ\downarrow$. But $r : M \to F(A)$ given by $r(m) = \{b \mid \phi(b) \leq m\}$ provides a right adjoint to q so we know that q is a suplattice homomorphism, and therefore $F(A)$ provides the correct universal properties. ■

Theorem 5 (dcpo presentations present) *For any dcpo presentation* $(P, R, ...)$, **dcpo**$\langle P$ *(qua poset)*$\mid R\rangle$ *is well defined.*

Proof. First note that the problem reduces to a proof of the existence of **dcpo** coequalizers since the ideal completion of any poset is the free dcpo on that poset. The relevant dcpo coequalizer is of $e_1, e_2 : R \rightrightarrows idl(P)$. Suplattice coequalizers certainly exist [JoyTie 84]: if M is a suplattice and R a subset of $M \times M$, then the set of R-coherent elements forms the coequalizer, where an $m \in M$ is R-coherent iff for every aRb it is the case that $a \leq m$ iff $b \leq m$. The coequalizing map can be verified to be that which sends any element to the meet of the R-coherent elements greater than it (this function defining a left adjoint to the inclusion of R-coherent elements into M).

To find the dcpo coequalizer of $f, g : A \rightrightarrows B$, the first step is to take the suplattice coequalizer of Ff and Fg, giving a suplattice homomorphism $h' : F(B) \to C'$. Here $F : \mathbf{dcpo} \to \mathbf{Sup}$ is the free functor (left adjoint to the forgetful functor) as in the previous lemma. Applying the forgetful functor and precomposing with the unit, we get a dcpo morphism $h'\circ\downarrow : B \to C'$. Next take the image factorization in **dcpo** to get $i \circ h : B \to C \to C'$. (The category of dcpos does have image factorizations: take the least sub-dcpo generated by the set-theoretic image of the function to be factorized.)

h is the required dcpo coequalizer of f and g. If $k : B \to D$ composes equally with f and g, then $F(k)$ factors via C' as $k' \circ h'$ (say). Because h is a cover and $\downarrow_D$ is monic, we get that k factors via h. (The pullback of D along $k' \circ i$ must be the whole of C.)

$$\begin{array}{ccc} B & \overset{h}{\twoheadrightarrow} & C \\ & & \downarrow i \\ k\downarrow & \swarrow & C' \\ & & \downarrow k' \\ D & \underset{\downarrow}{\hookrightarrow} & F(D) \end{array}$$

Uniqueness follows since h is epi. ■

Before proving the main double coverage theorem, we first prove a result that uses the techniques of the preframe coverage theorem [JoVic 91].

Proposition 6 *Let L be a join semilattice and R a join-stable set of directed relations on it. Then*

$$\mathbf{Sup}\langle L \text{ (qua } \vee\text{-SemiLat)} \mid R\rangle \cong \mathbf{dcpo}\langle L \text{ (qua poset)} \mid R\rangle.$$

Proof. By Theorem 5, the right-hand side is defined; let us denote it by A. Now for any $l \in L$ the join stability assumption on R enables us to define a dcpo morphism $\alpha_l : A \to A$ by

$$\alpha_l(l') = l' \vee l.$$

$\alpha_l(a)$ is an upper bound of l and a, and is monotone in l (as well as a dcpo morphism in a). Repeating this we can define, for each $a \in A$, a dcpo morphism $\beta_a : A \to A$ by

$$\beta_a(l) = \alpha_l(a).$$

Then $\beta_a(b)$, as a binary operation, is a dcpo morphism in both variables yielding an upper bound of a and b and has the idempotence property $\beta_a(a) = a$. From this we can deduce that $\beta_a(b)$ is the least upper bound of a and b and so A is a suplattice. (The nullary bound is easy: it is 0_L.)

It is now easy to show that A does indeed have the universal property required by the left-hand side. ■

Theorem 7 (Double Coverage Theorem) *If* $(L, R, ...)$ *is a DL-site, then*

$$\mathbf{Fr}\langle L \text{ (qua DL)} \mid R\rangle \cong \mathbf{dcpo}\langle L \text{ (qua poset)} \mid R\rangle$$

Proof. We have

$$\begin{aligned}
&\mathbf{Fr}\langle L \text{ (qua DL)} \mid R\rangle \\
&\qquad \cong \mathbf{Fr}\langle L \text{ (qua } \wedge\text{-SemiLat)} \mid \text{(qua } \vee\text{-SemiLat)}, R\rangle \\
&\qquad \cong \mathbf{Sup}\langle L \text{ (qua poset)} \mid \text{(qua } \vee\text{-SemiLat)}, R\rangle \\
&\qquad \cong \mathbf{Sup}\langle L \text{ (qua } \vee\text{-SemiLat)} \mid R\rangle.
\end{aligned}$$

where the middle step is an application of the original coverage theorem CovThm. The relations "qua ∨-SemiLat" and R are meet stable, the former by the distributivity of L and the latter by definition of DL-site. Finally, apply Proposition 6 to get the result. ■

Remark 8 *Given a DL-site* $(L, R, ...)$ *presenting* X*, we already know from the suplattice and preframe coverage theorems ([AbrVic 93], [JoVic 91]) that*

$$\begin{aligned}
\mathbf{Fr}\langle L \text{ (qua DL)} \mid R\rangle &\cong \mathbf{Sup}\langle L \text{ (qua } \vee\text{-SemiLat)} \mid R\rangle \\
&\cong \mathbf{PreFr}\langle L \text{ (qua } \wedge\text{-SemiLat)} \mid R\rangle.
\end{aligned}$$

Suppose then that the Double Coverage Theorem is used to define a dcpo morphism $q : \Omega X \to \Omega Y$ *from a monotone function* $q' : L \to \Omega Y$*. It follows that* q *is a suplattice homomorphism iff* q' *preserves finite joins, and a preframe homomorphism iff* q' *preserves finite meets.*

2.2 Dcpo presentations for product locales

It is well known that locale product is the same thing as suplattice tensor (e.g. III, 2, of [JoyTie 84]). With the Double Coverage Theorem we can now describe locale product via a dcpo presentation.

Proposition 9 *Suppose X and Y are locales with DL sites (L^X, R^X) and (L^Y, R^Y). Then*

$$\begin{aligned}\Omega(X \times Y) \cong \mathbf{dcpo}\langle L^X \otimes_{\vee\text{-}SemiLat} L^Y \ \textit{(qua poset)} \mid \\ \bigvee\nolimits^{\uparrow}_{t \in e_1(r)} (t \otimes b \vee u) = \bigvee\nolimits^{\uparrow}_{t \in e_2(r)} (t \otimes b \vee u) \\ (r \in R^X, b \in L^Y, u \in L^X \otimes_{\vee\text{-}SemiLat} L^Y) \\ \bigvee\nolimits^{\uparrow}_{t \in e_1(r)} (a \otimes t \vee u) = \bigvee\nolimits^{\uparrow}_{t \in e_2(r)} (a \otimes t \vee u) \\ (r \in R^Y, a \in L^X, u \in L^X \otimes_{\vee\text{-}SemiLat} L^Y) \rangle\end{aligned}$$

Proof. We have

$$\begin{aligned}\Omega(X \times Y) &\cong \mathbf{Fr}\langle L^X, L^Y \text{ (qua DLs)} | R^X, R^Y\rangle \\ &\cong \mathbf{Fr}\langle L^X, L^Y \text{ (qua } \wedge\text{-SemiLats)} | (L^X, L^Y \text{ qua } \vee\text{-SemiLats}), R^X, R^Y\rangle \\ &\cong \mathbf{Fr}\langle L^X \times L^Y \text{ (qua } \wedge\text{-SemiLat)} | \vee\text{-bilinearity}, R^X \otimes L^Y, L^X \otimes R^Y\rangle\end{aligned}$$

where $R^X \otimes L^Y$ denotes the set of relations

$$\bigvee\nolimits^{\uparrow}_{t \in e_1(r)} t \otimes b = \bigvee\nolimits^{\uparrow}_{t \in e_2(r)} t \otimes b$$

for $r \in R^X, b \in L^Y$ and similarly for $L^X \otimes R^Y$. For the moment, we are writing $t \otimes b$ in $L^X \times L^Y$ for the pair (t, b). Of course, this really becomes $t \otimes b$ when we map $L^X \times L^Y$ to $L^X \otimes L^Y$. We see that the relations obtained are all meet stable, and so

$$\begin{aligned}\Omega(X \times Y) &\cong \mathbf{Sup}\langle L^X \times L^Y \text{ (qua poset)} | \vee\text{-bilinearity}, R^X \otimes L^Y, L^X \otimes R^Y\rangle \\ &\cong \mathbf{Sup}\langle L^X \otimes_{\vee\text{-SemiLat}} L^Y \text{ (qua } \vee\text{-SemiLat)} | R^X \otimes L^Y, L^X \otimes R^Y\rangle\end{aligned}$$

Now we can make the relations join-stable by adding us as in the statement, and we can apply Proposition 6. ∎

Remark 10 *If $g : Y_1 \to Y_2$ is a locale map and X is a locale then (it can be checked that) the frame homomorphism $\Omega(g \times X) : \Omega(Y_2 \times X) \to \Omega(Y_1 \times X)$ is given by $\Omega(g \times X)(b \otimes a) = \Omega g(b) \otimes a$.*

2.3 Semilattice tensor via poset presentations

$\Omega(X \times Y)$ is therefore now described in terms of a dcpo presentation involving a semilattice tensor. That $L^X \otimes_{\vee\text{-SemiLat}} L^Y$ exists is immediate from universal algebra; but to exploit it in what follows we will have to be a bit more explicit about how it is constructed.

Proposition 11 *Let A and B be two join semilattices. Then their join semilattice tensor $(A \otimes_{\vee\text{-}SemiLat} B)$ is given by*

$$\mathbf{Pos}\langle \mathcal{F}(A \times B) \text{ (qua poset)} \mid \{(\bigvee_{i\in I} a_i, \bigvee_{j \in J} b_j)\} \cup U = \{(a_i, b_j) | i \in I, j \in J\} \cup U \qquad (U \in \mathcal{F}(A \times B))\rangle.$$

($\mathcal{F}$ here denotes the (Kuratowski) finite powerset; the relations are over all Kuratowski finite indexing sets I and J.)

Proof. Let C be the poset presented above, with universal monotone function $\gamma : \mathcal{F}(A \times B) \to C$ satisfying the relations. Because of the join stability of the relations, binary union on $\mathcal{F}(A \times B)$ defines a binary operation $\vee$ on C, $\gamma(U) \vee \gamma(V) = \gamma(U \cup V)$. This is binary join, and in fact C is a $\vee$-semilattice with γ a homomorphism. (The nullary join is $\gamma(\emptyset)$.)

Now suppose $\theta : A \times B \to D$ is bilinear for some join semilattice D. The mapping $U \longmapsto \bigvee_{(a,b)\in U} \theta(a, b)$ respects the relations that define C, since

$$\theta(\bigvee_{i\in I} a_i, \bigvee_{j\in J} b_j) \vee \bigvee_{(a,b)\in U} \theta(a, b) = \bigvee_{i\in I} \bigvee_{j \in J} \theta(a_i, b_j) \vee \bigvee_{(a,b)\in U} \theta(a, b)$$

The monotone map defined by this mapping clearly commutes with the construction of join on C and so there is a (necessarily unique) join semilattice from C to D extending θ. ■

Remark 12 *An exactly dual construction shows how to exhibit $A \otimes_{\wedge\text{-}SemiLat} B$. Similarly, the locale product can be presented as the dcpo generated by a meet-semilattice tensor qua poset. The proof of this fact (which is identical to that of Proposition 9 in structure, but with meets interchanged with joins) requires the preframe version of the coverage theorem.*

2.3.1 On a result of Fraser

It is well known that the coproduct of two commutative rings is just their tensor product, and an analogous result [Fraser 76] shows that the coproduct of two distributive lattices is a tensor product with respect to either the join semilattice tensor or, dually, the meet semilattice structure. One uses the universal property of the join tensor product to define the meets as a bilinear function, and similarly with joins and meets exchanged. A more general and abstract proof is given in [JoVic 91].

As an aside, in the case where (as indeed we shall have it) A and B are distributive lattices, it is actually possible to give an explicit description of the full preorder on $\mathcal{F}(A \times B)$: we have $\bigvee_{i=1}^{m} a_i \otimes b_i \leq \bigvee_{j=1}^{n} a'_j \otimes b'_j$ in the tensor product iff for every i there is some ϕ_i, a distributive lattice word in n variables, such that

$$a_i \leq \phi_i(\mathbf{a}') \text{ and } b_i \leq \widetilde{\phi}_i(\mathbf{b}').$$

(Here, $\phi_i(\mathbf{a}')$ is the term formed by substituing a'_j in the j^{th} place of the word ϕ_i for $j = 1, ..., n$. $\tilde{}$ is defined by interchanging meets and joins. More precisely, if L_n is the free distributive lattice on n generators, then $\tilde{} : L_n \to L_n^{op}$ is the lattice homomorphism that fixes the generators.) This was proved in [Fraser 76]. The classical proof there relies on the spatiality of spectral locales, but it is not hard to give a shorter and more direct constructive proof. The essential step in the argument is to show that if $\theta : A \times B \to C$ is a bilinear function to a join semilattice C, then the extension

$$\overline{\theta}(\bigvee_{i=1}^{m} a_i \otimes b_i) = \bigvee_{i=1}^{m} \theta(a_i, b_i)$$

is well defined. This relies on a lemma that for every word ϕ we have

$$\theta(\phi(\mathbf{a}), \widetilde{\phi}(\mathbf{b})) \leq \bigvee_{i=1}^{m} \theta(a_i, b_i),$$

and the constructive proof of this is by showing that the set of such words ϕ is closed under finite meets and joins and contains all the generators (variables).

3 The ideal completion as a locale

Definition 13 *Let P be a poset. The locale $Idl(P)$ is defined by*

$$\begin{aligned}\Omega Idl(P) = \mathbf{Fr}\langle \uparrow p \ (p \in P) | \\ &\uparrow p \leq \uparrow q \ (p \geq q) \\ &1 \leq \textstyle\bigvee_{p \in P} \uparrow p \\ &\uparrow p \wedge \uparrow q \leq \bigvee \{\uparrow r | p \leq r, q \leq r\} \quad \rangle\end{aligned}$$

The set of global points of $Idl(P)$ (that is the set of locale maps $1 \to Idl(P)$) is $idl(P)$. $\Omega Idl(P)$ is equivalent to the set of Scott opens on $idl(P)$. In fact, $Idl(P)$ is constructively spatial and there is a bijection between locale maps $Idl(P_1) \to Idl(P_2)$ and dcpo maps $idl(P_1) \to idl(P_2)$. If $e : idl(P_1) \to idl(P_2)$ is a dcpo map then the corresponding locale map $e' : Idl(P_1) \to Idl(P_2)$ is defined by $\Omega e'(\uparrow p_2) = \bigvee\{\uparrow p_1 \mid p_2 \in e(\downarrow p_1)\}$. See [Vickers 93] or Ch. 1 Sect. 1.6 of [Townsend 96] for further details. Note that if P is discrete then $Idl(P)$ is the discrete locale P.

Given a DL-site$(L, R, ...)$, the locale $Idl(L)$ will play an important role in our development, providing the connection between the frame-theoretic discussions of presentations and a more purely localic one. The following result is a fragment of a more general (and well known) topos-theoretic conclusion which states that if C is a small category and E and F are two toposes, then there is a bijection between functors $C \to \mathbf{Top}(E, F)$ and geometric morphisms $[C^{op}, \mathbf{Set}] \times E \to F$.

Proposition 14 *Let $(P, \leq)$ be a poset and X, Y two locales. Then the following are equivalent.*

1. *Monotone functions* $P \to \mathbf{Fr}(\Omega Y, \Omega X)$.

2. *dcpo maps* $idl(P) \to \mathbf{Fr}(\Omega Y, \Omega X)$.

3. *Locale maps* $Idl(P) \times X \to Y$.

Further: (a) the bijection (2)$\Longleftrightarrow$(3) is natural with respect to dcpo maps $idl(P_1) \to idl(P_2)$ *and (b) the bijections are also natural with respect to locale maps* $X_1 \to X_2$.

Note that the naturality proved here is what is needed to deal with the functions $R \to idl(L)$ (with the discrete order on R) that arise in DL-sites.

Proof. The equivalence between (1) and (2) is immediate since $idl(P)$ is the free dcpo on P qua poset. ($\mathbf{Fr}(\Omega Y, \Omega X)$ is always a dcpo, the directed joins being calculated elementwise – e.g. Lemma 1.11 of Ch. 2, [Johnstone 82].)

(2) $\Longleftrightarrow$(3). Given a monotone function $f : P \to \mathbf{Fr}(\Omega Y, \Omega X)$ define $F : Idl(P) \times X \to Y$ by

$$\Omega F(b) = \bigvee_{p \in P} \uparrow p \otimes f(p)(b).$$

Using the presentation of $Idl(P)$ one can check directly that this is a frame homomorphism.

Note that $\Omega(Idl(P) \times X) \cong \Omega Idl(P) \otimes_{\mathbf{Sup}} \Omega X$, i.e. the suplattice tensor description of locale product is used in this proof. In the other direction (given $F : Idl(P) \times X \to Y$), define f by $p \longmapsto \Omega(F \circ (\downarrow p \times X))$ where the point $\downarrow p : 1 \to Idl(P)$ is the ideal generated by p. Then $\Omega(\downarrow p \times X) : \Omega Idl(P) \otimes \Omega X \to \Omega X$ is given by

$$\uparrow q \otimes a \longmapsto \bigvee \{a \mid q \leq p\}.$$

(The join on the right is of a subsingleton set, with at most one element a, and that only if $q \leq p$.)

One can then show that the correspondence $f \leftrightarrow F$ is a bijection.

Naturality.

(a) Suppose we have a dcpo map $f : idl(P_1) \to idl(P_2)$. So f corresponds to $F : Idl(P_1) \to Idl(P_2)$ defined by

$$\Omega F(\uparrow q) = \bigvee \{\uparrow p | q \in f(\downarrow p)\}.$$

To put it another way, we have $q \in f(\downarrow p)$ iff $\uparrow p \leq \Omega F(\uparrow q)$. Then f acts on $\mathbf{Loc}(Idl(P_2) \times X, Y)$ by precomposition with $F \times X$. To prove naturality with respect to f suppose we have $g : idl(P_2) \to \mathbf{Fr}(\Omega Y, \Omega X)$ (corresponding to $G : Idl(P_2) \times X \to Y$). These give rise to composites

$$g \circ f : idl(P_1) \to idl(P_2) \to \mathbf{Fr}(\Omega Y, \Omega X)$$
$$G \circ (F \times X) : Idl(P_1) \times X \to Idl(P_2) \times X \to Y$$

and so to complete the proof we must show that these composites correspond under the bijection we have established between (2) and (3). Given the correspondence between (1) and (2) it is sufficient to calculate for each $b \in \Omega Y$,

$$\begin{aligned}(\Omega F \otimes \Omega X)(\bigvee_{q \in P_2} \uparrow q \otimes g(\downarrow q)(b)) &= \bigvee_{q \in P_2} \Omega F(\uparrow q) \otimes g(\downarrow q)(b) \\ &= \bigvee_{q \in P_2} (\bigvee \{\uparrow p | q \in f(\downarrow p)\} \otimes g(\downarrow q)(b)) \\ &= \bigvee_{q \in f(\downarrow p)} (\uparrow p \otimes g(\downarrow q)(b)) \\ &= \bigvee_{p \in P} (\uparrow p \otimes \bigvee \{g(\downarrow q)(b) | q \in f(\downarrow p)\}) \\ &= \bigvee_{p \in P} (\uparrow p \otimes g \circ f(\downarrow p)(b)).\end{aligned}$$

(Recall $\Omega(F \times X) \equiv \Omega F \otimes \Omega X$.) For $\bigvee \{g(\downarrow q)(b) | q \in f(\downarrow p)\} = g \circ f(\downarrow p)(b)$ simply note that $f(\downarrow p) = \bigvee^{\uparrow} \{\downarrow q \mid q \in f(\downarrow p)\}$.

(b) Naturality for locale maps $X_1 \to X_2$ is immediate from construction. ■

Corollary 15 *Let P be a set, and X and Y two locales. Then there is a bijection between functions $f : P \to \mathbf{Fr}(\Omega Y, \Omega X)$ and locale maps $F : P \times X \to Y$.*

Proof. P is discrete and so $Idl(P) \cong P$. (There is abuse of notation: P is a set and a discrete locale.) ■

4 Dcpo morphisms as locale maps

The results of the previous Section are now considered in conjunction with the Double Coverage Theorem 7, and this enables a localic characterization of dcpo morphisms between frames to be given.

Suppose we have a DL-site (L, R, e_1, e_2). Since $idl(L)$ is the discrete reflection of $Idl(L)$, the two functions $e_i : R \to idl(L)$ correspond to two maps $e'_i : R \to Idl(L)$.

Proposition 16 *Let (L, R, e_1, e_2) be a DL-site presenting a locale X, and let W be a locale. Then there is a bijection between dcpo morphisms $\Omega X \to \Omega W$ and maps $Idl(L) \times W \to \mathbb{S}$ that compose equally with the two maps $e'_i \times W : R \times W \to Idl(L) \times W$. Moreover this bijection is natural in W.*

Proof. Since $\Omega\mathbb{S}$ is free on one generator, ΩW is isomorphic to $\mathbf{Fr}(\Omega\mathbb{S}, \Omega W)$. By the Double Coverage Theorem (7), dcpo morphisms $\Omega X \to \mathbf{Fr}(\Omega\mathbb{S}, \Omega W)$ are equivalent to monotone functions $\alpha : L \to \mathbf{Fr}(\Omega\mathbb{S}, \Omega W)$ respecting the relations, and so to dcpo morphisms $\beta : idl(L) \to \mathbf{Fr}(\Omega\mathbb{S}, \Omega W)$ composing equally with the e_is. On the other hand, by Proposition 14 the monotone functions $\alpha : L \to \mathbf{Fr}(\Omega\mathbb{S}, \Omega W)$ are equivalent to maps $\overline{\overline{\alpha}} : Idl(L) \times W \to \mathbb{S}$. We must show that β composes equally with the e_is iff $\overline{\overline{\alpha}}$ composes equally with the maps $e'_i \times X$. This is a consequence of the naturality part of Proposition 14, with P_1 and P_2 specialized as R (with its discrete order) and L. ■

4.1 Weak exponentiation

To complete the proof of the main result we shall need to check that $\mathbb{S}^X$ exists weakly in **Loc**, and this is an interesting fact in itself. Recall (e.g. [CarbRos 00]) that the definition of weak exponentiation is the same as true exponentiation, but without the uniqueness requirement on the exponential transpose. In other words, Y^X exists weakly if there exists a locale W^{XY} and a map $ev : W^{XY} \times X \to Y$ such that for any map $z : Z \times X \to Y$ there exists a (not necessarily unique) map $\overline{z} : Z \to W^{YX}$ such that $ev \circ (\overline{z} \times X) = z$.

Proposition 17 *For any locale X presented by DL-site $(L^X, R^X, ...)$, the ideal completion locale $Idl(L^X)$ is a weak exponential $\mathbb{S}^X$.*

Proof. By Proposition 16 with $W = X$, the identity function on ΩX corresponds to a map $ev : Idl(L^X) \times X \to \mathbb{S}$ that composes equally with the maps $e'_i \times X : R \times X \to Idl(L^X) \times X$. This is the evaluation map. As an open in $Idl(L^X) \times X$, it is $\bigvee_{l \in L^X} \uparrow l \otimes l$.

Given $c : Y \times X \to \mathbb{S}$ define $\overline{c} : Y \to Idl(L^X)$ by

$$\Omega\overline{c}(\uparrow l) = \bigvee \{b \in \Omega Y \mid b \otimes l \leq c\}.$$

Then, as an open in $Y \times X$, $ev \circ (\overline{c} \times X)$ is

$$\begin{aligned} &\bigvee_{l \in L^X} (\bigvee \{b \in \Omega Y \mid b \otimes l \leq c\}) \otimes l \\ &\qquad = \bigvee \{b \otimes l | b \otimes l \leq c\} = c \end{aligned}$$

■

The weak exponential $\mathbb{S}^X$ can also be found via the spectrum $Spec(L^X)$ of L^X, whose frame is $idl(L^X)$. This locale is spectral and so is locally compact and hence exponentiable. But there is a locale inclusion $i : X \hookrightarrow Spec(L^X)$ and $\mathbb{S}$ is injective with respect to locale inclusions (in particular with respect to $Z \times i$ for any Z) and so $\mathbb{S}^{Spec(L^X)}$ is a weak exponential with weak evaluation map $ev \circ (Z \times i)$, where ev is the true evaluation map at $Spec(L^X)$. Acknowledgment is due to Martín Escardó for this description of the weak exponential. It can be verified that $\mathbb{S}^{Spec(L^X)} \cong Idl(L^X)$.

5 Dcpo morphisms as natural transformations

The results so far have, to use the language of the set-class distinction, concerned *sets*. (However, in Section 8.1 we shall qualify this in a topos-theoretic interpretation, and indeed we have taken care to use reasoning that is valid in the internal logic of toposes.) We now turn to issues that require more care regarding classes. **Loc** is a large category, and the presheaf category $[\mathbf{Loc}^{op}, \mathbf{Set}]$ cannot be assumed to have small hom-classes. Our main result in this section is to show that if X and W are locales, then the hom-class from $\mathbb{S}^X = \mathbf{Loc}(\_ \times X, \mathbb{S})$ to $\mathbb{S}^W = \mathbf{Loc}(\_ \times W, \mathbb{S})$ is in fact (in bijection with) a set.

Theorem 18 *Let X be a locale. Then there are bijections Φ_W, natural in locales W, between –*

- *natural transformations* $\mathbf{Loc}(\_\times X, \mathbb{S}) \to \mathbf{Loc}(\_\times W, \mathbb{S})$*, and*
- *dcpo morphisms* $\Omega X \to \Omega W$*.*

Proof. Suppose X is presented by a DL-site (L^X, R^X, e_1, e_2), and let $ev : Idl(L^X) \times X \to \mathbb{S}$ be the weak evaluation map.

Let $\alpha : \mathbf{Loc}(\_\times X, \mathbb{S}) \dot{\to} \mathbf{Loc}(\_\times W, \mathbb{S})$ be a natural transformation. Since ev composes equally with the maps $e_i' \times X$, it follows that $\alpha_{Idl(L^X)}(ev) : Idl(L^X) \times W \to \mathbb{S}$ composes equally with the maps $e_i' \times W$ and hence (by Proposition 16) corresponds to a dcpo morphism $\Omega X \to \Omega W$. Define $\Phi_W(\alpha)$ to be this dcpo morphism.

Now suppose we have $a : Y \times X \to \mathbb{S}$ with weak transpose $\overline{a} : Y \to Idl(L^X)$. We have $a = ev \circ (\overline{a} \times X)$, so, by naturality of α,

$$\alpha_Y(a) = \alpha_{Idl(L^X)}(ev) \circ (\overline{a} \times W)$$

Therefore, α is uniquely determined by $\Phi_W(\alpha)$.

In the other direction, say $q : \Omega X \to \Omega W$ is given as a dcpo morphism. To define $\alpha^q : \mathbf{Loc}(\_\times X, \mathbb{S}) \dot{\to} \mathbf{Loc}(\_\times W, \mathbb{S})$ we define, for every locale Y, a dcpo morphism $\Omega(Y \times X) \to \Omega(Y \times W)$. Suppose that Y is presented by a DL-site (L^Y, R^Y, e_1, e_2). If we write M for the finitary tensor product $L^Y \otimes_{\vee\text{-SemiLat}} L^X$, then Proposition 9 tells us that

$$\begin{aligned}
\Omega(Y \times X) \cong \mathbf{dcpo}\langle M \text{ (qua poset)} \mid \\
&{\bigvee}^{\uparrow}_{t \in e_1(r)} ((t \otimes a) \vee u) = {\bigvee}^{\uparrow}_{t \in e_2(r)} ((t \otimes a) \vee u) \\
&\qquad (r \in R^Y, a \in L^X, u \in M) \\
&{\bigvee}^{\uparrow}_{t \in e_1(r)} ((b \otimes t) \vee u) = {\bigvee}^{\uparrow}_{t \in e_2(r)} ((b \otimes t) \vee u) \\
&\qquad (r \in R^X, a \in L^Y, u \in M)\rangle.
\end{aligned}$$

We first define a monotone function $F : M \to \Omega(Y \times W)$ via a function $F' : \mathcal{F}(L^Y \times L^X) \to \Omega(Y \times W)$,

$$F'(U) = \bigvee_{U' \in \mathcal{F}U} (\textstyle\bigwedge_{(b,a) \in U'} b \otimes q(\bigvee_{(b,a) \in U'} a)),$$

and using Proposition 11. Note that if this F is well defined and induces a dcpo map $\Omega(Y \times X) \to \Omega(Y \times W)$ for every locale Y then naturality in Y is easy to verify given remark 10. The relations of Proposition 11 can be checked as follows; on the left hand side,

$$\begin{aligned}
&F'(\{(\bigvee_{i\in I} b_i, \bigvee_{j\in J} a_j)\} \cup U) \\
&= \bigvee_{U'\in\mathcal{F}U} (\bigwedge_{(b,a)\in U'} b \otimes q(\bigvee_{(b,a)\in U'} a) \\
&\qquad \vee ((\bigvee_{i\in I} b_i) \wedge \bigwedge_{(b,a)\in U'} b) \otimes q(\bigvee_{j\in J} a_j \vee \bigvee_{(b,a)\in U'} a)) \\
&= \bigvee_{U'\in\mathcal{F}U} (\bigwedge_{(b,a)\in U'} b \otimes q(\bigvee_{(b,a)\in U'} a) \\
&\qquad \vee \bigvee_{i\in I} (b_i \wedge \bigwedge_{(b,a)\in U'} b) \otimes q(\bigvee_{j\in J} a_j \vee \bigvee_{(b,a)\in U'} a)) \\
&= F'(U) \vee \bigvee_{U'\in\mathcal{F}U} \bigvee_{i\in I} (b_i \wedge \bigwedge_{(b,a)\in U'} b) \otimes q(\bigvee_{j\in J} a_j \vee \bigvee_{(b,a)\in U'} a)
\end{aligned}$$

while on the right,

$$\begin{aligned}
&F'(\{(a_i, b_j) | i \in I, j \in J\} \cup U) \\
&\qquad = \bigvee_{U'\in\mathcal{F}U} \bigvee_{K\in\mathcal{F}(I\times J)} (\bigwedge_{(i,j)\in K} b_i \wedge \bigwedge_{(b,a)\in U'} b) \otimes q(\bigvee_{(i,j)\in K} a_j \vee \bigvee_{(b,a)\in U'} a)
\end{aligned}$$

First we show that LHS $\leq$ RHS. By taking $K = \emptyset$, we get $F'(U) \leq$ RHS. For a disjunct on the left with U' and i, we take the same U' on the right and $K = \{i\} \times J$.

Next we show RHS $\leq$ LHS. Consider a disjunct on the right with U' and K. For Kuratowski finite sets, emptiness is a decidable property (see e.g. [JohLin 78]), so we can argue by cases for K empty or inhabited. If K is empty, then the disjunct $\leq F'(U)$. On the other hand, suppose $(i', j') \in K$. Then

$$\text{disjunct } \leq (b_{i'} \wedge \bigwedge_{(b,a)\in U'} b) \otimes q(\bigvee_{j\in J} a_j \vee \bigvee_{(b,a)\in U'} a) \leq \text{ LHS.}$$

Therefore F' does factor through $M = L^Y \otimes_{\vee\text{-SemiLat}} L^X$ and so defines a monotone function $F : M \to \Omega(Y \times W)$. It must next be checked that F respects the relations of the dcpo presentation of $\Omega(Y \times X)$.

$$\begin{aligned}
&F(b \otimes a \vee \bigvee_{i=1}^{m} b_i \otimes a_i) \\
&= \bigvee_{I'\in\mathcal{F}\{1,\ldots,m\}} (\bigwedge_{i\in I'} b_i \otimes q(\bigvee_{i\in I'} a_i) \vee (b \wedge \bigwedge_{i\in I'} b_i) \otimes q(a \vee \bigvee_{i\in I'} a_i))
\end{aligned}$$

This right-hand expression preserves directed joins in b (treated as an element of ΩY), and it follows (substituting t for b) that F respects the first family of relations. Similarly, it preserves directed joins in a and so F respects the second family of relations.

It can be shown that $\Phi_W(\alpha^q) = q$, by comparing their corresponding elements in $\mathbf{Loc}(Idl(L^X) \times W, \mathbb{S})$. On the one hand q gives us the element

$\bigvee_{l \in L} \uparrow l \otimes q(l)$. On the other, $\alpha^q_{Idl(L^X)}$ gives us

$$\begin{aligned}
&\alpha^q_{Idl(L^X)}(\bigvee_{l \in L^X} \uparrow l \otimes l) \\
&\qquad = \bigvee^{\uparrow}_{L' \in \mathcal{F} L^X} \bigvee_{L'' \in \mathcal{F} L'} (\bigwedge_{l \in L''} \uparrow l \otimes q(\bigvee_{l \in L''} l)) \\
&\qquad = \bigvee^{\uparrow}_{L' \in \mathcal{F} L^X} \bigvee_{L'' \in \mathcal{F} L'} (\uparrow (\bigvee L'') \otimes q(\bigvee L'')) \\
&\qquad = \bigvee_{l \in L} \uparrow l \otimes q(l).
\end{aligned}$$

Naturality in W is clear from the statement of Proposition 16. ■

Note that the $\otimes_{\wedge\text{-SemiLat}}$ could have been used instead in the last part of this proof. A description of the bijection in terms of the $\otimes_{\wedge\text{-SemiLat}}$ is used when specializing to the upper power locale (Theorem 21).

6 The main results

Given the characterization of dcpo morphisms in terms of natural transformations, the main result is immediate:

Theorem 19 *If X is a locale then the exponential $\mathbb{S}^{\mathbb{S}^X}$ exists in $[\mathbf{Loc}^{op}, \mathbf{Set}]$ and is naturally isomorphic to the representable functor $\mathbf{Loc}(\_, \mathbb{P}X)$.*

Proof. $\Omega\mathbb{P}X$ is the free frame on ΩX qua dcpo. But the dcpo morphisms $\Omega X \to \Omega W$ have been characterized, naturally in W, as the natural transformations $\mathbb{S}^X \to \mathbb{S}^W$, i.e. exactly the natural transformations $\mathbf{Loc}(\_, W) \times \mathbb{S}^X \to \mathbf{Loc}(\_, \mathbb{S})$ (by the definition of the exponential $\mathbb{S}^W$ in $[\mathbf{Loc}^{op}, \mathbf{Set}]$). This set is exactly $\mathbb{S}^{\mathbb{S}^X}(W)$ and therefore it has been shown that $\mathbb{S}^{\mathbb{S}^X}(W) \cong \mathbf{Loc}(W, \mathbb{P}X)$ naturally in W. ■

6.1 The upper and lower powerlocales

The main result specializes to the upper and lower powerlocale constructions (P_U and P_L).

First note [Vickers 02] that any $\mathbb{P}X$ is an internal distributive lattice in $\mathbf{Loc}$, and hence (because the Yoneda embedding preserves finite limits) in $[\mathbf{Loc}^{op}, \mathbf{Set}]$. In particular this includes $\mathbb{S}$, which is $\mathbb{P}\emptyset$. It follows (because $Y \mapsto Y^X$ preserves all limits, being right adjoint to $Z \mapsto Z \times X$) that $\mathbb{S}^X$ is an internal distributive lattice in $[\mathbf{Loc}^{op}, \mathbf{Set}]$ for any locale X. The lattice structure on each component $\mathbf{Loc}(Y \times X, \mathbb{S})$ is inherited straightforwardly from the localic lattice structure of $\mathbb{S}$. Note also that if M and N are two internal lattices (or indeed internal algebras of any kind) in $[\mathbf{Loc}^{op}, \mathbf{Set}]$, then a morphism $\alpha : M \to N$ is a homomorphism iff every component $\alpha_X : M(X) \to N(X)$ is a homomorphism.

We shall also need the fact that if L is a distributive lattice, then $Idl(L)$ is an internal distributive lattice in $\mathbf{Loc}$. This follows because Idl provides a functor from **Pos** to **Loc** that preserves products, $Idl(P \times Q)$ being homeomorphic to $Idl(P) \times Idl(Q)$ by $\uparrow(p,q) \longleftrightarrow \uparrow p \otimes \uparrow q$. On monotone functions $f : P \to Q$, the functor Idl acts by $\Omega Idl(f)(\uparrow q) = \bigvee\{\uparrow p | q \leq f(p)\}$, and this enables us to calculate the inverse image functions for meet and join on $Idl(L)$:

$$\Omega(\wedge)(\uparrow l) = \bigvee\{\uparrow m \otimes \uparrow n) | l \leq m \wedge n\} = \uparrow l \otimes \uparrow l$$
$$\Omega(\vee)(\uparrow l) = \bigvee\{\uparrow m \otimes \uparrow n | l \leq m \vee n\}.$$

Since $\mathbb{S}$ is $Idl(\{\bot, \top\}, \bot \leq \top)$, we can use this to calculate inverse image functions for meet and join on $\mathbb{S}$. Expressing them as opens of $\mathbb{S} \times \mathbb{S}$, meet is $\uparrow\top \otimes \uparrow\top$ and join is $\uparrow\top \otimes 1 \vee 1 \otimes \uparrow\top = \uparrow\top ✠ \uparrow\top$ where $(a,b) \mapsto a ✠ b$ is the universal preframe bilinear map ([JoVic 91]; there ✠ is written as an upside down &). Note that $\uparrow\top$ is the free generator of $\Omega\mathbb{S}$.

Lemma 20 *Let L be a distributive lattice and W a locale. There is a bijection between monotone functions $f : L \to \Omega W$ and maps $F : Idl(L) \times W \to \mathbb{S}$. Then f preserves finite meets (respectively joins) iff F preserves finite meets (respectively joins) on $Idl(L)$.*

Proof. As explained in Proposition 16, the bijection is a consequence of Proposition 14. F, considered as an open of $Idl(L) \times W$, is $\bigvee_{l \in L} \uparrow l \otimes f(l)$.

Preservation by F of n-ary meets or joins on $Idl(L)$ means equality of two maps $Idl(L^n) \times W \to \mathbb{S}$. We shall present the argument for binary meets and joins. For binary meets, the first map is

$$F \circ (Idl(\wedge_L) \times W) : Idl(L^2) \times W \to Idl(L) \times W \to \mathbb{S}.$$

The second,

$$\wedge_{\mathbb{S}} \circ F^2 \circ \langle \pi_1 \times W, \pi_2 \times W \rangle \circ \cong :$$
$$Idl(L^2) \times W \cong Idl(L)^2 \times W \to (Idl(L) \times W)^2 \to \mathbb{S}^2 \to \mathbb{S},$$

can be expressed using the lattice operations in $\mathbf{Loc}(Idl(L^2) \times W, \mathbb{S})$ as

$$(F \circ (Idl(\pi_1) \times W)) \wedge (F \circ (Idl(\pi_2) \times W)).$$

(We are writing π_i for the product projections.) Writing $\uparrow\top$ for the generator of $\Omega\mathbb{S}$, we find the inverse image for $F \circ (Idl(\wedge) \times W)$ takes

$$\begin{aligned} \uparrow\top &\mapsto \bigvee_l \uparrow l \otimes f(l) \\ &\mapsto \bigvee_l \bigvee\{\uparrow(m,n) | l \leq m \wedge n\} \otimes f(l) \\ &= \bigvee_{mn} \uparrow(m,n) \otimes f(m \wedge n) \end{aligned}$$

which corresponds to the function $(m,n) \mapsto f(m \wedge n)$.

The inverse image for $\bigwedge_{i=1}^{2}(F \circ (Idl(\pi_i) \times W))$ takes

$$
\begin{aligned}
\uparrow\top &\mapsto \uparrow\top \otimes \uparrow\top \\
&\mapsto \bigvee_{mn} \uparrow m \otimes f(m) \otimes \uparrow n \otimes f(n) \\
&\mapsto \bigvee_{mn} \uparrow m \otimes \uparrow n \otimes f(m) \wedge f(n) \\
&\mapsto \bigvee_{mn} \uparrow (m,n) \otimes f(m) \wedge f(n)
\end{aligned}
$$

which corresponds to the function $(m,n) \mapsto f(m) \wedge f(n)$ from L^2 to ΩW.

It follows that F preserves binary meets in $Idl(L)$ iff f preserves binary meets.

For joins, we find the first map takes

$$
\begin{aligned}
\uparrow\top &\mapsto \bigvee_{l} \uparrow l \otimes f(l) \\
&\mapsto \bigvee_{l} \bigvee\{\uparrow (m,n) | l \leq m \vee n\} \otimes f(l) \\
&= \bigvee_{mn} \uparrow (m,n) \otimes f(m \vee n)
\end{aligned}
$$

which corresponds to the function $(m,n) \mapsto f(m \vee n)$. The second map takes

$$
\begin{aligned}
\uparrow\top &\mapsto \uparrow\top \otimes 1 \vee 1 \otimes \uparrow\top \\
&\mapsto \bigvee_{l} (\uparrow l \otimes f(l) \otimes \uparrow 0 \otimes 1) \vee \bigvee_{l} (\uparrow 0 \otimes 1 \otimes \uparrow l \otimes f(l)) \\
&\mapsto \bigvee_{l} ((\uparrow l \otimes \uparrow 0 \otimes f(l)) \vee (\uparrow 0 \otimes \uparrow l \otimes f(l))) \\
&\mapsto \bigvee_{l} (\uparrow (l,0) \vee \uparrow (0,l)) \otimes f(l) \\
&= \bigvee_{mn} \uparrow (m,n) \otimes f(m) \vee f(n)
\end{aligned}
$$

which corresponds to the function $(m,n) \mapsto f(m) \vee f(n)$. ■

Theorem 21 *Let X be a locale.*

1. *There is a bijection, natural in W, between locale maps $W \to P_L(X)$ and join semilattice homomorphisms $\mathbb{S}^X \to \mathbb{S}^W$.*
2. *There is a bijection, natural in W, between locale maps $W \to P_U(X)$ and meet semilattice homomorphisms $\mathbb{S}^X \to \mathbb{S}^W$.*
3. *There is a bijection, natural in W, between locale maps $W \to X$ and lattice homomorphisms $\mathbb{S}^X \to \mathbb{S}^W$.*

Proof. (1) In the context of Theorem 18 we just need to show that a map $W \to \mathbb{P}X$ factors via $P_L X$ (i.e. its dcpo morphism q between the frames preserves finite joins) iff the corresponding natural transformation $\alpha : \mathbb{S}^X \to \mathbb{S}^W$ is a join semilattice homomorphism.

First, suppose q preserves finite joins (so it is a suplattice homomorphism). Then the dcpo morphism of Theorem 18, from $\Omega(Y \times X)$ to $\Omega(Y \times W)$, assigns

$$\begin{aligned}
\bigvee_{i \in I} b_i \otimes a_i \longmapsto & \bigvee_{I' \in \mathcal{F} I} \bigwedge_{i \in I'} b_i \otimes q(\bigvee_{i \in I'} a_i) \\
&= \bigvee_{I' \in \mathcal{F} I} \bigwedge_{i \in I'} b_i \otimes \bigvee_{i \in I'} q(a_i) \\
&= \bigvee_{i \in I} b_i \otimes q(a_i)
\end{aligned}$$

and hence preserves finite joins. We can now combine Propositions 9 and 6 to deduce that our morphism $\Omega(Y \times X) \to \Omega(Y \times W)$ is a suplattice homomorphism, and so the corresponding function $\mathbb{S}^X(Y) \to \mathbb{S}^W(Y)$ preserves finite joins.

Now suppose that we are given a join semilattice homomorphism $\alpha : \mathbb{S}^X \to \mathbb{S}^W$. We suppose as usual that X is presented by a DL-site $(L, R, ...)$. By Remark 8 it suffices to show that the composite function $L \to \Omega X \to \Omega W$ preserves finite joins, and then by Lemma 20 it suffices to show that $\alpha_{Idl(L)}(ev)$ preserves finite joins in $Idl(L)$. Lemma 20 already tells us that ev does this, because it corresponds to the identity morphism on ΩX. For n-ary joins, we have

$$\begin{aligned}
\alpha_{Idl(L)}(ev) \circ (Idl(\vee) \times W) &= \alpha_{Idl(L^n)}(ev \circ (Idl(\vee) \times X)) \\
&= \alpha_{Idl(L^n)}(\bigvee_{i=1}^{n} (ev \circ (Idl(\pi_i) \times X))) \\
&= \bigvee_{i=1}^{n} \alpha_{Idl(L^n)}(ev \circ (Idl(\pi_i) \times X)) \\
&= \bigvee_{i=1}^{n} \alpha_{Idl(L)}(ev) \circ (Idl(\pi_i) \times W)
\end{aligned}$$

as required.

(2) The reverse direction – that if α preserves finite meets then so does q – follows by the same argument, replacing joins by meets. The forward direction requires a little more care. The problem is that in the presentation of Proposition 9 the relations are not meet stable, so it does not trivially give a preframe presentation. Instead we argue by duality as follows.

The theory of tensor products for join semilattices dualizes, giving tensor products for meet semilattices. If A and B are distributive lattices then their two tensor products are isomorphic, since both give the distributive lattice coproduct. However, the universal bilinear functions are different, that for the meet semilattice tensor being $a \maltese b = a \otimes 1 \vee 1 \otimes b$.

Proposition 9 now has a dual form using the meet semilattice tensor, and Theorem 18 has a dual proof in which, given q, $\beta^q_Y : \Omega(Y \times X) \to \Omega(Y \times W)$ is defined by

$$\beta^q_Y(\bigwedge_{(b,a) \in U} b \maltese a) = \bigwedge_{U' \in \mathcal{F} U} (\bigvee_{(b,a) \in U'} b) \maltese q(\bigwedge_{(b,a) \in U'} a).$$

It is not evident that this gives the same dcpo morphisms as the join semilattice definition, though we conjecture that it does. However, it gives the same natural transformation $\mathbf{Loc}(\_ \times X, \mathbb{S}) \to \mathbf{Loc}(\_ \times W, \mathbb{S})$ because the dual proof shows that it too gives back the original q. Now, just as in part (1), we can show that if q preserves finite meets then so does each β_Y^q, and hence so does the natural transformation.

(3) Follows by combining the first two parts. ■

7 Applications

7.1 The Strength of the Double Power Monad

As an application, the monad structure on $\mathbb{P}$ can be found fairly easily using this representation as $\mathbb{S}^{\mathbb{S}^X}$ (see e.g. [Taylor 02]). In particular, the strength $\chi : \mathbb{P}X \times Y \to \mathbb{P}(X \times Y)$ becomes $\chi : \mathbb{S}^{\mathbb{S}^X} \times Y \to \mathbb{S}^{\mathbb{S}^{X \times Y}}$ and can be defined by a λ-term

$$\chi(\Phi, y) = \lambda U.\ \Phi(\lambda x.\ U(x, y))$$

Defining the strength direct from the definition of $\mathbb{P}$ is a little intricate, and in fact seems to embody some of the argument of Theorem 18.

7.2 The localic reflection of $\mathbb{S}^X$

As a further application of the methods given here, we show that even though $\mathbb{S}^X$ is not always a locale (because X is not always exponentiable), it nonetheless has a localic reflection.

Proposition 22 *If ΩX and ΩY are two frames, then $\mathbf{dcpo}(\Omega Y, \Omega X)$ is a frame.*

Proof. Conceptually this is because $\mathbb{P}Y$ is a localic distributive lattice, so $\mathbf{Loc}(X, \mathbb{P}Y)$ is a distributive lattice as well as (by the dcpo-enrichment of $\mathbf{Loc}$) a dcpo. Reasoning internally it is easy enough to check that the finite meets and joins (calculated pointwise) of dcpo morphisms between frames are still dcpo morphisms. ■

Proposition 23 *Let X be a locale. Then the presheaf $\mathbb{S}^X = \mathbf{Loc}(\_ \times X, \mathbb{S})$ has a localic reflection Y. It is defined by*

$$\Omega Y = \mathbf{dcpo}(\Omega X, \Omega),$$

in other words the topology on Y is the Scott topology on the frame ΩX.

Proof. From the proof of Theorem 18 we see that if $q : \Omega X \to \Omega$ is a dcpo morphism, then we get a dcpo morphism $\alpha_W^q : \Omega(W \times X) \to \Omega W$. We therefore get a function $\alpha_W^- : \mathbf{dcpo}(\Omega X, \Omega) = \Omega Y \to \mathbf{dcpo}(\Omega(W \times X), \Omega W)$, and in fact this is a frame homomorphism. It follows that for every W we have a dcpo morphism $\Omega(W \times X) \to \mathbf{Fr}(\Omega Y, \Omega W)$, natural in W, and hence a natural transformation $\gamma : \mathbf{Loc}(\_ \times X, \mathbb{S}) \to \mathbf{Loc}(\_, Y)$, i.e. from $\mathbb{S}^X$ to Y.

Now suppose we have a natural transformation $\beta : \mathbb{S}^X \to Z$ for some locale Z. Again applying the argument of Theorem 18, using $\beta_{Idl(L^X)}$, we get a map $Idl(L^X) \to Z$ composing equally with the two maps from R^X. This gives us a dcpo morphism $\Omega X \to \mathbf{Fr}(\Omega Z, \Omega)$ and hence by Proposition 14 a frame homomorphism $\Omega Z \to \mathbf{dcpo}(\Omega X, \Omega) = \Omega Y$, so a locale map $\overline{\beta} : Y \to Z$. We find $\beta = \gamma; \overline{\beta}$, and in fact $\overline{\beta}$ is the unique such locale map. ■

8 Conclusions

We have shown how **Loc** can be embedded in a category ($[\mathbf{Loc}^{op}, \mathbf{Set}]$) in which $\mathbb{P}X \cong \mathbb{S}^{\mathbb{S}^X}$. This characterizes $\mathbb{P}X$ (and the other powerlocales too) in a way that depends purely on the categorical structure of **Loc**, not on the concrete structure of frames. At the same time we have also displayed techniques for calculating with $\mathbb{P}X$ that depend on presentation rather than on having the entire frame. It is our hope that this will prove useful in developing locale theory in contexts (such as formal topology within the doctrine of predicative type theory) where frames cannot be constructed as sets.

We hope also that the work will provide insight into the problem of axiomatizing a synthetic locale theory (see e.g. [Vickers 02]). For instance, an abstract category of spaces could be defined as an order enriched category **C** with an internal distributive lattice $\mathbb{S}$ such that $\mathbb{S}^{\mathbb{S}^X}$ exists for any space X. Using the techniques of Theorem 21 the familiar theory of the upper and lower power spaces re-emerges from a single assumption about the existence of a double power space. This is a subject for further work.

8.1 Remark on set-theoretic foundations

We have concealed some topos-theoretic aspects in the exposition, though they have influenced the mathematics in a number of places. In the initial sections (to Section 4), we have reasoned using topos-valid mathematics so that "set" can mean "object in a given topos". From Section 5 there arises the deeper question of external vs. internal sets and this is best understood by reference to Theorem 18. The theorem is stated as though there is simply a (not necessarily classical) category of sets in which we can discuss frames and hence also locales. The proof, however, is designed to yield a more subtle result about locales over toposes. Suppose S is an elementary topos (we believe our proofs do not require a natural number object) and $f : X \to S$ and $g : W \to S$ are two localic geometric morphisms, in other words locales over S. By the known correspondence [JoyTie 84] between locales and frames, we have two frames $\Omega^S(X_f)$ and $\Omega^S(W_g)$, internal in S. (The notation X_f denotes X, considered as a locale over S by the morphism f.) They can be calculated as $f_*(\Omega_X)$ and $g_*(\Omega_W)$. The known correspondence shows that locale maps $W_g \to X_f$, i.e.

geometric morphisms $W \to X$ making the triangle to S commute, correspond to morphisms $\Omega^S(X_f) \to \Omega^S(W_g)$ that are, internally, frame homomorphisms. What we show is that internal dcpo morphisms $\Omega^S(X_f) \to \Omega^S(W_g)$ are in bijection with natural transformations

$$\mathbf{Loc}/S(\_ \times_S X_f, \mathbb{S}_S) \to \mathbf{Loc}/S(\_ \times_S W_g, \mathbb{S}_S)$$

where $\mathbb{S}_S$ denotes the Sierpiński locale over S. Thus we have a correspondence not only between Scott continuity and naturality, but also between internal and external.

This has some effects on the shape of the proofs. Where the exposition refers to $\mathbf{Loc}(X, \mathbb{S})$ one might imagine this to be identical (or at least isomorphic) to the frame ΩX. However, in a more sophisticated interpretation, $\mathbf{Loc}(X, \mathbb{S})$, the set of locale maps from X to $\mathbb{S}$, is actually the set of global elements of ΩX. For any morphism $\Omega X \to \Omega W$ (i.e. $\Omega^S(X_f) \to \Omega^S(W_g)$) we can find a corresponding function $\mathbf{Loc}(X, \mathbb{S}) \to \mathbf{Loc}(W, \mathbb{S})$ (i.e. $\mathbf{Loc}/S(X_f, \mathbb{S}_S) \to \mathbf{Loc}/S(W_g, \mathbb{S}_S)$) by restricting to global elements, but we cannot necessarily go in the reverse direction. In a couple of places (one in the proof of Theorem 18 and more substantial ones in Theorem 21), a more direct proof can by found by using the component $\alpha_1 : \mathbf{Loc}(X, \mathbb{S}) \to \mathbf{Loc}(W, \mathbb{S})$ of a natural transformation as giving directly the morphism $q : \Omega X \to \Omega W$. In our broader context this is invalid and instead we carry out more explicit calculations using $\alpha_{Idl(L^X)}$.

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