Georgian Math. J., Ahead of Print
DOI 10.1515/gmj-2012-0011

Aspects of slice stability in locale theory

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Abstract. It is shown that a particular categorical axiomatisation of the category of locales is slice stable. This localic slice stability can be used to recover the fundamental theorem of topos theory. A categorical account of the ideal completion of a preorder is developed and is used to give a new proof of Joyal and Tierney's result on the slice stability of locales.

Keywords. Locale, power monad, ideal completion, sheaf, local homeomorphism, relational composition.

2010 Mathematics Subject Classification. 06D22, 18B25, 18B10, 18D20, 18F15.

1 Introduction

An axiom, placed on a category $\mathcal{C}$, is said to be slice stable if whenever it is satisfied by $\mathcal{C}$ it is also satisfied by the slice category $\mathcal{C}/X$ for every object X of $\mathcal{C}$. The easiest example is the axiom of having finite limits (i.e. being cartesian); if $\mathcal{C}$ has finite limits then so does $\mathcal{C}/X$ for any object X of $\mathcal{C}$. A non-example is the axiom of having finite products since there are examples of categories that have finite products but not equalizers. The fundamental theorem of topos theory, e.g. [4, A2.3], is the assertion that the axioms of an elementary topos are slice stable.

The category of locales, Loc, enjoys the property

$$\mathsf{Loc}_{\mathrm{Sh}(Y)} \simeq \mathsf{Loc}/Y$$

for any locale Y, where $\mathrm{Sh}(Y)$ is the topos of sheaves over Y, see [6]. So any fact about the category of locales that can be proved within a topos is true of any slice of the category of locales. Since most of the results of locale theory are provable using only intuitionistic logic and so can be proved in any topos, this applies to most of locale theory. From this we intuitively know that the category of locales is slice stable ('logically' slice stable say). Slice stability in the literature invariably refers to this logical slice stability. However what is missing is a set of axioms for the category of locales which is slice stable. Whilst axiomatic approaches have been developed (see [8–10]; and also [7, 12] for not unrelated techniques) these approaches are not shown to be slice stable. The main result of this paper is to show that the axiomatic accounts of locale theory developed in [8–10] are all slice stable with only trivial modifications (these trivial modifications do not affect the results available and so do not constitute a weakening of the theory).

Two applications are given. Firstly, we prove the fundamental theorem of topos theory as a consequence of this slice stability. Secondly, we exploit slice stability to prove $\mathsf{Loc}_{\mathrm{Sh}(Y)} \simeq \mathsf{Loc}/Y$, so showing that the familiar logical slice stability of locale theory is an aspect of axiomatic slice stability. We are able to prove this second result with no significant use of topos theoretic machinery as we do not have to develop set theoretic constructions internal to a topos of sheaves. However we will assume familiarity with the well understood relationship

$$\mathrm{Sh}(Y) \simeq \mathsf{LH}/Y$$

which is central to sheaf theory (here LH/Y is the category of local homeomorphisms over Y). The hope therefore is to make Joyal and Tierney's result on the slice stability of locales more accessible to sheaf theorists (albeit at the expense of a fairly involved discussion of various categorical aspects of locale theory).

This paper does not focus on the relationship between the categorical axioms discussed and their canonical model (the category locales). Therefore it does not provide much detail on the general topological motivation that underpins the work; however such motivation can be found in the literature, e.g. [3, 11]. For motivation on categorical approaches to topology/locale theory consult for example [7, 12].

1.1 Outline contents

We begin by making some comments on lattices internal to order enriched categories and recalling categorical change of base. This change of base technique can be described under mild categorical assumptions and is central to nearly all of the proofs in the rest of the paper.

Next we recall the axioms to be discussed and show that they are slice stable. The key technical result is a proof that the axiomatic representation of the double coverage theorem (introduced in [14]) is slice stable. The discussion then focuses on the power monads and we show that certain properties of these monads are, in the presence of the other axioms, slice stable. This allows us to conclude that all of the results of [8, 9] are slice stable.

In particular, we can then conclude that the Hofmann–Mislove theorem (a categorical account of which is the main result of [9]) is slice stable. As an application of this we prove axiomatically the known result that discrete objects are exponentiable and show how the isomorphism implicit in this observation can be described using relational composition. With this categorical description of relational composition we are able to introduce axiomatically the ideal completion of a preorder. We check that this construction has the right properties and that it agrees with the more general construction given in [9]. When acting on semilattices, we show that the ideal completion construction is functorial. Although the results contained within this section are just basic aspects of lattice theory the section is long. This is because we are no longer working within set theory but are reliant on a categorical axiomatisation and so every step is a categorical manipulation rather than a more familiar set theoretic one.

The next section consists of applications. It gives two proofs using the techniques developed; firstly, of the fundamental theorem of topos theory and, secondly, of the Joyal and Tierney result on slice stability (i.e. $\mathsf{Loc}_{\mathrm{Sh}(Y)} \simeq \mathsf{Loc}/Y$). The fundamental theorem of topos theory becomes a categorical triviality since every topos, $\mathcal{E}$, is equivalent to the category of discrete locales over $\mathcal{E}$.

The proof of $\mathsf{Loc}_{\mathrm{Sh}(Y)} \simeq \mathsf{Loc}/Y$ is a bit more involved. Firstly, we note that the category $\mathrm{Sh}(Y)$ can be expressed axiomatically using local homeomorphisms in the usual manner. Next, the proof centres on the verification that, in the slice over any locale Y, the ideal completion of local homeomorphisms (that are also internal semilattices) has a right adjoint which, in fact, is monadic. The proof is then completed by verifying that the induced monad is the downset monad whose algebras are exactly frames in $\mathrm{Sh}(Y)$.

The final section is a summary section which also contains a table of the axioms together with some comments on them, for example noting known dependencies. An appendix is included that contains a proof that a particular axiom (used only in [10] and not used for the results of this paper) is slice stable.

2 Categorical background

2.1 Order enriched categories

Our category $\mathcal{C}$, to be axiomatised, is order enriched. Let us start by making a few general comments about order enriched categories. Firstly, it must be noted that when limits or any other universal constructions are discussed relative to an order enriched category it is assumed that they are order enriched; that is, establish order isomorphisms on the relevant partially ordered homsets rather than just bijections. For example if $\mathcal{C}$ has binary products then it is assumed that the function

$$\mathcal{C}(Z, X \times Y) \xrightarrow{(\pi_1 \circ (\_), \pi_2 \circ (\_))} \mathcal{C}(Z, X) \times \mathcal{C}(Z, Y)$$

is an order isomorphism and not just a bijection.

In an order enriched category with finite products we will invariably discuss *order* internal meet (or join) semilattices. These are just ordinary internal semilattices but with the further requirement that the join (meet) map be left (right) adjoint to the diagonal and that the unit be left (right) adjoint to the nullary diagonal (i.e. the unique map to 1). Therefore for an object X in an order enriched category we have that there is at most one order enriched join or meet semilattice structure on it; this is by the uniqueness of adjoints.

An order enriched functor $F : \mathcal{C} \to \mathcal{D}$ between the order enriched categories $\mathcal{C}$ and $\mathcal{D}$ is just an ordinary functor but with the additional property that F, on morphisms, is monotone. To prove that F preserves order internal join or meet semilattices it is sufficient to check that it preserves products. Note that for any particular order internal join or meet semilattice, X say, it just needs to be checked that F preserves $\Delta_X : X \to X \times X$ and $!^X : X \to 1$ in order to conclude that $F(X)$ is an order internal join or meet semilattice.

Let us comment on the presheaf categories of order enriched categories. Central to the axiomatic system to be discussed is the interaction between $\mathcal{C}$ and its presheaf category $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$. However if $\mathcal{C}$ is order enriched then $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ does not have a natural ordering on all of its homsets. There are two ways round this problem: (i) develop the theory relative to the order enriched category $[\mathcal{C}^{\mathrm{op}}, \mathsf{Pos}]$ where Pos is the category of posets or (ii) note that in practice for all the presheaves that we are concerned with the homsets between them do in fact have a natural ordering and so are posets. The first option is viable and in fact all the axioms do hold with $[\mathcal{C}^{\mathrm{op}}, \mathsf{Pos}]$ in place of $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$. However we have chosen to stick with the more familiar $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ for this exposition as this avoids the need to re-check the axioms in the slightly different context of an order enriched presheaf category.

Finally we make some observations about internal semilattices in $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$. Say X and Y are objects of an order enriched cartesian category $\mathcal{C}$. Consider the presheaf

$$\mathcal{C}^{\mathrm{op}} \to \mathsf{Set}, \quad Z \mapsto \mathcal{C}(Z \times Y, X).$$

The notation X^Y is used for this presheaf since it can be verified that X^Y is the exponential $\mathcal{C}(\_, X)^{\mathcal{C}(\_, Y)}$ in the category $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$. If X is an order internal join semilattice in $\mathcal{C}$ then X^Y is an internal join semilattice in $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$. Now the homsets $[X^Y, X^Y \times X^Y]$, $[X^Y \times X^Y, X^Y]$, $[1, X^Y]$ and $[X^Y, 1]$ are all posets as they inherit an ordering from $\mathcal{C}$ and it can be seen that the binary and nullary join maps on X^Y are left adjoint to the binary and nullary diagonal maps in $[X^Y, X^Y \times X^Y]$ and $[X^Y, 1]$ respectively. Therefore to prove that a functor $F : [\mathcal{C}^{\mathrm{op}}, \mathsf{Set}] \to [\mathcal{D}^{\mathrm{op}}, \mathsf{Set}]$ preserves the semilattice structure on X^Y we have but to check that it preserves the ordering on these homsets and the diagonal maps. We will apply this reasoning below to specialise our categorical change of base result which is the subject of the next subsection.

2.2 Categorical change of base

The work below is going to rely on categorical change of base, see [8]. Since the main lemma relevant to this can be stated under very mild categorical assumptions and is easy to prove, it seems appropriate to start with this aspect. We first assume an order enriched cartesian category $\mathcal{C}$ with some distinguished object $\mathbb{S}$.

Let X be an object of $\mathcal{C}$. We use $\mathbb{S}_X$ to denote $\pi_2 : \mathbb{S} \times X \to X$, an object of the slice $\mathcal{C}/X$. Then,

Lemma 2.2.1. *For any morphism $f : X \to Y$ of $\mathcal{C}$, define*

$$f^\# : [(\mathcal{C}/Y)^{\mathrm{op}}, \mathsf{Set}] \to [(\mathcal{C}/X)^{\mathrm{op}}, \mathsf{Set}]$$

by precomposition with Σ_f and define

$$f_* : [(\mathcal{C}/X)^{\mathrm{op}}, \mathsf{Set}] \to [(\mathcal{C}/Y)^{\mathrm{op}}, \mathsf{Set}]$$

by precomposition with f^*. Then $f^\# \dashv f_*$ and for any objects A, B of $\mathcal{C}/X$, $\mathcal{C}/Y$ respectively we have

$$f^\# \mathbb{S}_Y^B \cong \mathbb{S}_X^{f^*B}$$

and

$$f_* \mathbb{S}_X^A \cong \mathbb{S}_Y^{\Sigma_f A}.$$

Further there is a natural order isomorphism between the sets of natural transformations,

$$[\mathbb{S}_X^{f^*B}, \mathbb{S}_X^A] \cong [\mathbb{S}_Y^B, \mathbb{S}_Y^{\Sigma_f A}],$$

ordered pointwise in the obvious manner.

Here the standard notation for the pullback adjunction

$$\Sigma_f \dashv f^* : \mathcal{C}/X \rightleftarrows \mathcal{C}/Y$$

is being used.

Proof. To see that $f^\# \dashv f_*$ observe that the unit and counit of $\Sigma_f \dashv f^*$ can be used to define natural transformations $f^\# f_* F \to F$ for each $F : (\mathcal{C}/X)^{\mathrm{op}} \to \mathsf{Set}$ and $f_* f^\# G \to G$ for each $G : (\mathcal{C}/Y)^{\mathrm{op}} \to \mathsf{Set}$. But the induced natural transformations $f^\# f_* \to \mathrm{Id}$ and $f_* f^\# \to \mathrm{Id}$ can be seen to satisfy the triangular identities for $f^\#$ and f_* (since the counit and unit of $\Sigma_f \dashv f^*$ satisfy the triangular identities for f^* and Σ_f) and so $f^\# \dashv f_*$.

Both $f^\# \mathbb{S}_Y^B \cong \mathbb{S}_X^{f^*B}$ and $f_* \mathbb{S}_X^A \cong \mathbb{S}_Y^{\Sigma_f A}$ are immediate since

$$\mathcal{C}/Y(B \times \Sigma_f A, \mathbb{S}_Y) \cong \mathcal{C}/X(f^*B \times A, \mathbb{S}_X)$$

holds for every A in $\mathcal{C}/X$ and every B in $\mathcal{C}/Y$. □

The following lemma will be used in our main technical result on the slice stability of Axiom 4 (to follow). It is essentially an exercise in the definition of $f^\#$ and f_* just given.

Lemma 2.2.2. *For any $W \xrightarrow{l} Y$, a morphism of an order enriched cartesian category $\mathcal{C}$ with some distinguished object $\mathbb{S}$, and for any natural transformation $\alpha : \mathbb{S}_Y^A \to \mathbb{S}_Y^B$ with objects A and B of $\mathcal{C}/Y$ and any object X of $\mathcal{C}$ we have that*

$$[W_* l^\#(\alpha)]_X = \alpha_{\Sigma_l W^* X}$$

and

$$\alpha_{W_l} = [W_* l^\#(\alpha)]_1.$$

Note we are using W_l as notation for the object $l : W \to Y$ of $\mathcal{C}/Y$. Further note that we are, of course, using the notation Σ_W and W^* for the pullback adjunction $\mathcal{C}/W \rightleftarrows \mathcal{C}$ and are using $W^\# \dashv W_*$ (rather than $(!^W)^\# \dashv !^W_*$) for the extension to natural transformations.

Proof. In summary, the proof is about unwinding the definition of the extensions of the functors Σ_W and l^* to natural transformations.

Firstly, consider some natural transformation $\gamma : \mathbb{S}_W^C \to \mathbb{S}_W^D$, then $[W_*(\gamma)]_X$ for any object X of $\mathcal{C}$ is defined by

$$\mathcal{C}(X \times \Sigma_W(C), \mathbb{S}) \cong \mathcal{C}/W(W^*X \times_W C, \mathbb{S}_W)$$
$$\xrightarrow{\gamma_{W^*X}} \mathcal{C}/W(W^*X \times_W D, \mathbb{S}_W) \cong \mathcal{C}(X \times \Sigma_W(D), \mathbb{S}).$$

That is, $[W_*(\gamma)]_X = \gamma_{W^*X}$.

Secondly, for $\alpha : \mathbb{S}_Y^A \to \mathbb{S}_Y^B$ and for any object E of $\mathcal{C}/W$, $[l^\#(\alpha)]_E$ is defined by

$$\mathcal{C}/W(E \times_W l^*A, \mathbb{S}_W) \cong \mathcal{C}/Y(\Sigma_l E \times_Y A, \mathbb{S}_Y)$$
$$\xrightarrow{\alpha_{\Sigma_l E}} \mathcal{C}/Y(\Sigma_l E \times_Y B, \mathbb{S}_Y) \cong \mathcal{C}/W(E \times_W l^*B, \mathbb{S}_W).$$

That is, $[l^\#(\alpha)]_E = \alpha_{\Sigma_l E}$. Combining these we get the first claim. For the second claim note that $\Sigma_l W^* 1 = W_l$. □

We now discuss a sharpening of Lemma 2.2.1 to situations where the additional assumption is made that $\mathbb{S}$ is an order internal join semilattice. Certainly then $\mathbb{S}_Y^{\Sigma_f(\_)}$ transforms $\mathbb{S}_X^A$ into an order internal join semilattice in $[\mathcal{C}/Y^{\mathrm{op}}, \mathsf{Set}]$ as the lemma shows it is (a restriction of) a right adjoint. We would like to conclude, moving in the opposite direction, that $\mathbb{S}_X^{f^*(\_)}$ transforms any $\mathbb{S}_Y^B$ into a join semilattice in $[\mathcal{C}/X^{\mathrm{op}}, \mathsf{Set}]$. If $h^* : \mathcal{C}/W \to \mathcal{C}/Z$ preserves finite coproduct (for every morphism $h : Z \to W$) then this is the case since then

$$\mathbb{S}_Y^B \times \mathbb{S}_Y^B \cong \mathbb{S}_Y^{B+B} \quad \text{and} \quad \mathbb{S}_X^{f^*(B+B)} \cong \mathbb{S}_X^{f^*B+f^*B} \cong \mathbb{S}_Y^{f^*B} \times \mathbb{S}_Y^{f^*B},$$

showing that $\mathbb{S}_X^{f^*(\_)}$ preserves binary product (nullary product is similar). In summary,

Proposition 2.2.3. *If $\mathcal{C}$ is a cartesian order enriched category with finite coproducts such that pullback preserves finite coproducts then for any order internal join semilattice $\mathbb{S}$, any $f : X \to Y$ and any A and B objects of $\mathcal{C}/X$ and $\mathcal{C}/Y$, respectively,*

(i) $\mathbb{S}_Y^{\Sigma_f(\_)}$ *takes* $\sqcup_{\mathbb{S}_X^A}$ *to* $\sqcup_{\mathbb{S}_Y^{\Sigma_f A}}$ *(and similarly nullary join),*

(ii) $\mathbb{S}_X^{f^*(\_)}$ *takes* $\sqcup_{\mathbb{S}_Y^B}$ *to* $\sqcup_{\mathbb{S}_X^{f^*B}}$ *(and similarly nullary join), and*

(iii) *the order isomorphism between natural transformations of Lemma* 2.2.1 *specialises to natural transformations that are join semilattice homomorphisms.*

Proof. (i) and (ii) are covered in the preamble. For (iii) note that the unit and counit of the adjunction in the lemma are

$$\mathbb{S}_Y^B \xrightarrow{\mathbb{S}_Y^{\epsilon_B}} \mathbb{S}_Y^{\Sigma_f f^* B} \quad \text{and} \quad \mathbb{S}_X^{f^*\Sigma_f A} \xrightarrow{\mathbb{S}_X^{\eta_A}} \mathbb{S}_X^A,$$

respectively, where η and ϵ are the unit and counit of the adjunction $\Sigma_f \dashv f^*$. But both these maps are internal join semilattice homomorphisms and so (iii) is clear as (i) and (ii) show that $\mathbb{S}_X^{f^*(\_)}$ and $\mathbb{S}_Y^{\Sigma_f(\_)}$, on morphisms, preserve the property of being an internal join semilattice homomorphism. □

Of course an identical discussion could have been had about order internal meet semilattices.

3 The axioms

3.1 Basic axioms

We now state various axioms that can be placed on a category $\mathcal{C}$ and show that they are slice stable. They are all stable under the order enrichment which means that the theory developed has an implicit order enriched duality. The axioms are all true when $\mathcal{C} = \mathsf{Loc}$ and are used in [8,9] to develop various aspects of locale theory.

Axiom 1. $\mathcal{C}$ is an order enriched category with finite limits and finite colimits.

It has been commented on above that the property of being cartesian is slice stable. It is similarly trivial that the property of being cocartesian is slice stable.

Axiom 2. For any objects A, B and C in $\mathcal{C}/Z$, $A \times (B + C) \cong A \times B + A \times C$. Further $A \times 0 \cong 0$.

This axiom is about all slices and can be seen to be a slice stable axiom almost by definition. It is equivalent to the assertion that for any morphism $f : X \to Y$ of $\mathcal{C}$, the pullback functor $f^* : \mathcal{C}/X \to \mathcal{C}/Y$ preserves finite coproducts. (I have not been able to exhibit a category that is distributive for which this axiom fails. Although it is surely known whether distributivity is sufficient for Axiom 2, it has not proved possible here to establish this either way.)

Axiom 3. There is an order internal distributive lattice, denoted by $\mathbb{S}$, such that given a pullback

$$\begin{array}{ccc} a^*(i) & \longrightarrow & 1 \\ \downarrow & & \downarrow i \\ X & \xrightarrow{\;a\;} & \mathbb{S} \end{array}$$

a is uniquely determined by $a^*(i) \hookrightarrow X$ for $i : 1 \to \mathbb{S}$ equal to either $0_{\mathbb{S}}$ or $1_{\mathbb{S}}$.

Such an $\mathbb{S}$ is called a *Sierpiński object*. If $\mathbb{S}$ is a Sierpiński object in $\mathcal{C}$ then for any object Z of $\mathcal{C}$ it is easy to verify that $\mathbb{S}_Z$ is a Sierpiński object of $\mathcal{C}/Z$, so the axiom is slice stable. It can be shown (see [8]) that this axiom implies, for any object Z of $\mathcal{C}$ and any morphism $\alpha : \mathbb{S}^Z \to \mathbb{S}$,

(i) $\sqcap_{\mathbb{S}}(\alpha \times \mathrm{Id}_{\mathbb{S}}) \sqsubseteq \alpha \sqcap_{\mathbb{S}^Z} (\mathrm{Id}_{\mathbb{S}^Z} \times \mathbb{S}^{!^Z})$, and

(ii) $\alpha \sqcup_{\mathbb{S}^Z} (\mathrm{Id}_{\mathbb{S}^Z} \times \mathbb{S}^{!^Z}) \sqsubseteq \sqcup_{\mathbb{S}}(\alpha \times \mathrm{Id}_{\mathbb{S}})$,

where $\sqsubseteq$ is the order enrichment on $\mathcal{C}$.

It is these last two inequalities which are sufficient to develop certain aspects of locale theory and so are taken as axioms in [9] rather than our Axiom 3.

Capturing the usual localic notion we have:

Definition 3.1.1. A monomorphism $X_0 \hookrightarrow X$ is an *open subobject* if it is the pullback of $1_{\mathbb{S}} : 1 \to \mathbb{S}$ and it is a *closed subobject* if it is the pullback of $0_{\mathbb{S}} : 1 \to \mathbb{S}$.

It is shown in [8] that this axiom establishes an order isomorphism between the open subobjects of X and morphisms $X \to \mathbb{S}$; in particular to prove that $a \sqsubseteq b$ for $a, b : X \to \mathbb{S}$ it is sufficient to prove that for any morphism $Z \xrightarrow{p} X$ if p factors via a^*1 then p factors via b^*1.

Although in locale theory the Sierpiński locale can be defined categorically (for example as the categorical tensor $1 \otimes \mathbf{2}$, where $\mathbf{2}$ is the external poset $\{0 \leq 1\}$) it is not the case that Axiom 3 determines an object unique up to isomorphism. To see this note that the terminal object always satisfies Axiom 3. However this lack of uniqueness makes no difference to the development of the theory to follow.

The next axiom is a categorical interpretation of the content of the double coverage theorem introduced in [14].

Axiom 4. For any equalizer diagram

$$E \xhookrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

in $\mathcal{C}$ the diagram

$$\mathbb{S}^X \times \mathbb{S}^X \times \mathbb{S}^Y \underset{\sqcap(1\times\sqcup)(\mathrm{Id}\times\mathrm{Id}\times\mathbb{S}^g)}{\overset{\sqcap(1\times\sqcup)(\mathrm{Id}\times\mathrm{Id}\times\mathbb{S}^f)}{\rightrightarrows}} \mathbb{S}^X \xrightarrow{\mathbb{S}^e} \mathbb{S}^E$$

is a coequalizer in the full subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ consisting of all objects of the form $\mathbb{S}^W$.

Here $\sqcap(1 \times \sqcup)$ is the composite

$$\mathbb{S}^X \times \mathbb{S}^X \times \mathbb{S}^X \xrightarrow{1\times\sqcup} \mathbb{S}^X \times \mathbb{S}^X \xrightarrow{\sqcap} \mathbb{S}^X.$$

Our main technical result for this paper is the proof that this axiom is slice stable.

Proposition 3.1.2. *Assuming Axioms* 1–3*, Axiom* 4 *is slice stable.*

Proof. Say

$$E \xhookrightarrow{e} X_1 \underset{h_2}{\overset{h_1}{\rightrightarrows}} X_2$$

is an equalizer diagram in $\mathcal{C}/Y$, we must show that for any $\alpha : \mathbb{S}_Y^{X_1} \to \mathbb{S}_Y^Z$ for which

$$\alpha \sqcap (1 \times \sqcup)(\mathrm{Id} \times \mathrm{Id} \times \mathbb{S}_Y^{h_1}) = \alpha \sqcap (1 \times \sqcup)(\mathrm{Id} \times \mathrm{Id} \times \mathbb{S}_Y^{h_2}) \tag{1}$$

there exists unique $\beta : \mathbb{S}_Y^E \to \mathbb{S}_Y^Z$ such that $\beta \mathbb{S}_Y^e = \alpha$. The first thing to note is that by application of the change of base proposition above we can assume that $Z = 1$ (recall that change of base preserves Sierpiński meet and join; Proposition 2.2.3).

So say we are given $\alpha : \mathbb{S}_Y^{X_1} \to \mathbb{S}_Y$ satisfying (1). To define $\beta : \mathbb{S}_Y^E \to \mathbb{S}_Y$ we must, for every $l : W \to Y$, define a map

$$\mathcal{C}/Y(E \times_Y W_l, \mathbb{S}_Y) \to \mathcal{C}/Y(W_l, \mathbb{S}_Y).$$

Since $\mathcal{C}/Y(E \times_Y W_l, \mathbb{S}_Y) \cong \mathcal{C}(E \times_Y W, \mathbb{S})$ and $\mathcal{C}/Y(W_l, \mathbb{S}_Y) \cong \mathcal{C}(W, \mathbb{S})$ this amounts to defining a map

$$\mathcal{C}(E \times_Y W, \mathbb{S}) \to \mathcal{C}(W, \mathbb{S})$$

for each $l : W \to Y$. Now if α satisfies (1) then $W_* l^\#(\alpha)$ satisfies

$$\begin{aligned}(W_* l^\#(\alpha)) &\sqcap (1 \times \sqcup)(\mathrm{Id} \times \mathrm{Id} \times \mathbb{S}^{h_1 \times \mathrm{Id}}) \\ &= (W_* l^\#(\alpha)) \sqcap (1 \times \sqcup)(\mathrm{Id} \times \mathrm{Id} \times \mathbb{S}^{h_2 \times \mathrm{Id}}),\end{aligned}$$

since, again by change of base, the extended functors W_* and $l^\#$ preserve the Sierpiński meet and join. Therefore, by Axiom 4, there exists a unique natural transformation $\gamma^{W_l} : \mathbb{S}^{E \times_Y W} \to \mathbb{S}^W$ such that $\gamma^{W_l} \mathbb{S}^{e \times \mathrm{Id}} = W_* l^\#(\alpha)$.

We define $\beta : \mathbb{S}_Y^E \to \mathbb{S}_Y$ by $\beta_{W_l} \equiv [\gamma^{W_l}]_1$. The construction of β from α is monotone, so to complete the proof it remains to verify

A. that β is natural,

B. $\beta \mathbb{S}_Y^e = \alpha$, and

C. if $\delta \mathbb{S}_Y^e = \alpha$ for some other natural transformation $\delta : \mathbb{S}_Y^E \to \mathbb{S}_Y$ then $\delta = \beta$.

Proof of A. Say $n : W_l \to V_m$ is a morphism in $\mathcal{C}/Y$ then by the first part of Lemma 2.2.2 the square

$$\begin{array}{ccc} \mathbb{S}^{X_1 \times_Y V} & \xrightarrow{V_* m^\#(\alpha)} & \mathbb{S}^V \\ \Big\downarrow{\scriptstyle \mathbb{S}^{\mathrm{Id} \times n}} & & \Big\downarrow{\scriptstyle \mathbb{S}^n} \\ \mathbb{S}^{X_1 \times_Y W} & \xrightarrow{W_* l^\#(\alpha)} & \mathbb{S}^W \end{array}$$

commutes by the naturality of α. However $V_* m^\#(\alpha)$ factors as $\gamma^{V_m} \mathbb{S}^{e \times \mathrm{Id}}$ and $W_* l^\#(\alpha)$ factors as $\gamma^{W_l} \mathbb{S}^{e \times \mathrm{Id}}$ and since $\mathbb{S}^{e \times \mathrm{Id}}$ is an epimorphism (Axiom 4) we can conclude that

$$\begin{array}{ccc} \mathbb{S}^{E\times_Y V} & \xrightarrow{\gamma^{V n}} & \mathbb{S}^{V} \\ {\scriptstyle \mathbb{S}^{\mathrm{Id}\times n}}\downarrow & & \downarrow{\scriptstyle \mathbb{S}^{n}} \\ \mathbb{S}^{E\times_Y W} & \xrightarrow{\gamma^{W_l}} & \mathbb{S}^{W} \end{array}$$

commutes. By applying these natural transformations at 1 we therefore obtain the fact that β is natural.

Proof of B. By Lemma 2.2.2, for any W_l,

$$\alpha_{W_l} = [W_* l^\#(\alpha)]_1 = [\gamma^{W_l}\mathbb{S}^{e\times\mathrm{Id}}]_1 = [\gamma^{W_l}]_1[\mathbb{S}^{e\times\mathrm{Id}}]_1 = \beta_{W_l}[\mathbb{S}^e_Y]_{W_l}.$$

Proof of C. For such a δ, for any W_l, $W_* l^\#(\alpha) = W_* l^\#(\delta)\mathbb{S}^{e\times\mathrm{Id}}$, by applying $W_* l^\#$ to the triangle $\delta\mathbb{S}^e_Y = \alpha$. It follows that $W_* l^\#(\delta) = \gamma^{W_l}$ by the uniqueness part of Axiom 4. But by the lemma $\delta_{W_l} = [W_* l^\#(\delta)]_1 = [\gamma^{W_l}]_1 = \beta_{W_l}$ and so we are done. □

Axiom 5. The map $\mathbb{S}^{(\_)} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ reflects isomorphisms.

Given Axiom 4, Axiom 5 implies that $\mathbb{S}^{(\_)}$ is faithful. To see this note that if g_1 and g_2 are two morphisms with $\mathbb{S}^{g_1} = \mathbb{S}^{g_2}$ then $\mathbb{S}^e$ is an isomorphism by Axiom 4, where e is the equalizer of g_1 and g_2. This is sufficient to show that $g_1 = g_2$ as e is therefore an isomorphism in the presence of Axiom 5.

For the slice stability of Axiom 5, say $\mathbb{S}^f_Y : \mathbb{S}^{W_l}_Y \to \mathbb{S}^{V_m}_Y$ is an isomorphism in $[(\mathcal{C}/Y)^{\mathrm{op}}, \mathsf{Set}]$ for some morphism f of $\mathcal{C}/Y$. Then $\Sigma_Y(\mathbb{S}^f_Y)$, i.e. $\mathbb{S}^f : \mathbb{S}^W \to \mathbb{S}^V$, is an isomorphism in $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ and so f is an isomorphism in $\mathcal{C}$ by the axiom. Therefore there exists $g : W \to V$ such that $gf = \mathrm{Id}_V$ and $fg = \mathrm{Id}_W$. It must just be checked that g is a morphism of $\mathcal{C}/Y$, i.e. that $mg = l$. As f is a morphism of $\mathcal{C}/Y$ we have $lf = m$. Therefore $mg = lfg = l$ and we are done.

In [9] a (not necessarily strict) strengthening of Axiom 5 is exploited so, for completeness, this is included here.

Axiom 5′. Any internal distributive lattice homomorphism $\alpha : \mathbb{S}^Y \to \mathbb{S}^X$ is of the form $\mathbb{S}^f$ for unique $f : X \to Y$.

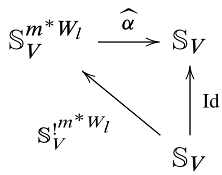
Proposition 3.1.3. *Given Axioms* 1–4*, Axiom* 5′ *is slice stable.*

Proof. Say $\alpha : \mathbb{S}^{W_l}_Y \to \mathbb{S}^{V_m}_Y$ is an internal distributive lattice homomorphism. By change base to V, we obtain $\widehat{\alpha} : \mathbb{S}^{m^* W_l}_V \to \mathbb{S}_V$, the adjoint transpose of α, which is also an internal distributive lattice homomorphism. Now by Axiom 3 applied to $\mathcal{C}/V$, the relations

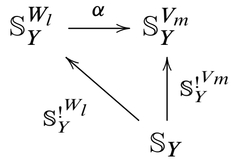
(i) $\sqcap(\widehat{\alpha}\times 1)\sqsubseteq\widehat{\alpha}\sqcap(1\times\mathbb{S}_V^{!m^*W_l})$, and

(ii) $\widehat{\alpha}\sqcup(1\times\mathbb{S}_V^{!m^*W_l})\sqsubseteq\sqcup(\widehat{\alpha}\times 1)$

hold and since $\widehat{\alpha}$ preserves 0 and 1, this shows that $\widehat{\alpha}$ is split by $\mathbb{S}^!_V$; in other words, the diagram



commutes. By taking the adjoint transpose (i.e. changing the base) back to Y, we get that



commutes. Then by applying the functor Σ_Y to this triangle we see both that $\Sigma_Y(\alpha)$, since it is a distributive lattice homomorphism, is $\mathbb{S}^f$ for some $f : V \to W$ and also, by the uniqueness part of Axiom 5′, that $lf = m$. This shows that f is a morphism of $\mathcal{C}/Y$. To prove that $\alpha = \mathbb{S}_Y^f$ it is sufficient to check that

$$\alpha\mathbb{S}_Y^{(\mathrm{Id},f)} = \mathbb{S}_Y^f\mathbb{S}_Y^{(\mathrm{Id},f)} \qquad (*)$$

since (Id, f) is a regular monomorphism (apply Axiom 4). But $(*)$ follows since the adjoint transpose (back to 1) of both sides is $\Sigma_Y(\alpha)$. □

Axiom 6. For any regular epimorphism $q : X \twoheadrightarrow Q$ in $\mathcal{C}$, for any $\alpha, \beta : \mathbb{S}^Y \to \mathbb{S}^Q$, if $\mathbb{S}^q\alpha = \mathbb{S}^q\beta$ then $\alpha = \beta$.

In other words, if q is a regular epimorphism then $\mathbb{S}^q$ is a monomorphism in the full subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ consisting of all objects of the form $\mathbb{S}^Y$. Note that in [8] this aspect of the axiomatisation is presented as:

> "The image of a coequalizer diagram under $\mathbb{S}^{(\_)} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ is an equalizer diagram provided we restrict to the full category of all objects in $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ of the form $\mathbb{S}^X$."

However, in application it is only the fact that $\mathbb{S}^q$ is a monomorphism that is used, so all the results of [8] are still available with the weakening just given.

Proposition 3.1.4. *Assuming Axioms* 1–4*, Axiom* 6 *is slice stable.*

Proof. Say $X \stackrel{q}{\to} Q$ is a regular epimorphism in $\mathcal{C}/Y$ and that we are given α, $\beta : \mathbb{S}_Y^{W_l} \to \mathbb{S}_Y^{Q}$ with $\mathbb{S}_Y^q \alpha = \mathbb{S}_Y^q \beta$ for any object W_l of $\mathcal{C}/Y$. Since W_l exists as an equalizer

$$W_l \xrightarrow{(\mathrm{Id},l)} W_Y \underset{\Delta\pi_2}{\overset{l\times\mathrm{Id}}{\rightrightarrows}} Y_Y$$

in $\mathcal{C}/Y$, to prove that $\alpha = \beta$ it is sufficient to prove that $\alpha\mathbb{S}_Y^{(\mathrm{Id},l)} = \beta\mathbb{S}_Y^{(\mathrm{Id},l)}$ and appeal to Axiom 4. But the adjoint transpose of $\alpha\mathbb{S}_Y^{(\mathrm{Id},l)}$ under change of base back to 1 is $\Sigma_Y(\alpha)$ and the adjoint transpose of $\beta\mathbb{S}_Y^{(\mathrm{Id},l)}$ is $\Sigma_Y(\beta)$; so it remains to prove $\Sigma_Y(\alpha) = \Sigma_Y(\beta)$, which is trivial by the application of Axiom 6 to q as q is certainly a regular epimorphism in $\mathcal{C}$ (as coequalizers in $\mathcal{C}/Y$ are created in $\mathcal{C}$). □

In summary, we have therefore now shown that all of the axiomatic account of weak triquotient assignments in locale theory, which is the main topic of [8], is automatically slice stable. The slice stable phrasing of the axioms in that paper is redundant.

3.2 The double power object axiom

To complete this discussion it is useful to point out that various other categorical axioms for locales are also automatically slice stable. The most important axiom is probably the next one, which offers a categorical characterization of the double power object.

Axiom 7. The functor $\mathbb{S}^{(\_)} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$, once its codomain is restricted to the full category of all objects in $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ of the form $\mathbb{S}^X$, has a right adjoint.

Such a right adjoint, it can be verified, sends $\mathbb{S}^X$ to $\mathbb{S}^{\mathbb{S}^X}$, i.e. the double exponential. Conversely, if such an exponential exists and is representable then the axiom is satisfied. The notation $\mathbb{P}(X) = \mathbb{S}^{\mathbb{S}^X}$ is used and it can be verified that a monad is defined on $\mathcal{C}$ using the universal properties of exponentiation. With this axiom we can stop writing 'the full subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ consisting of all objects of the form $\mathbb{S}^X$', and write $\mathcal{C}_{\mathbb{P}}^{\mathrm{op}}$, since this full subcategory is readily seen to be (weakly) equivalent to the opposite of the Kleisli category. Note that Axiom 4 implies that $\mathbb{P}$ takes regular monomorphisms to regular monomorphisms. This is not to say that it preserves equalizer diagrams, though it can be checked that it preserves coreflexive equalizers.

It is worth noting further that this axiom is a (not necessarily strict) strengthening of Axiom 6.

Proposition 3.2.1. *Given Axioms* 1–4*, Axiom* 7 *is slice stable.*

Proof. Firstly, notice that for any objects X and Y of $\mathcal{C}$, the double exponential $\mathbb{S}_Y^{\mathbb{S}_Y^{X_Y}}$ exists in $[(\mathcal{C}/Y)^{\mathrm{op}}, \mathsf{Set}]$. It is given by $\mathbb{P}(X)_Y$. This can be verified by change of base since, for any object W_l of $\mathcal{C}/Y$,

$$\mathcal{C}/Y(W_l, \mathbb{P}(X)_Y) \cong \mathcal{C}(W, \mathbb{P}(X)) \cong \mathrm{Nat}[\mathbb{S}^X, \mathbb{S}^W] \cong \mathrm{Nat}[\mathbb{S}_Y^{X_Y}, \mathbb{S}_Y^{W_l}],$$

where the last step is by change of base.

Now as in the previous proof, any X_f, an object of $\mathcal{C}/Y$, occurs as an equalizer

$$X_f \xhookrightarrow{(\mathrm{Id}, f)} X_Y \underset{\Delta\pi_2}{\overset{f\times \mathrm{Id}}{\rightrightarrows}} Y_Y$$

and this gives rise, via Axiom 4 in the slice $\mathcal{C}/Y$, to a coequalizer in (a full subcategory of) $[(\mathcal{C}/Y)^{\mathrm{op}}, \mathsf{Set}]$ which we can write as

$$\mathbb{S}_Y^{(X+X+Y)_Y} \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \mathbb{S}_Y^{X_Y} \xrightarrow{\mathbb{S}_Y^{(\mathrm{Id}, f)}} \mathbb{S}_Y^{X_f}.$$

If we therefore define $\mathbb{P}_Y(X_f)$ to be the equalizer of

$$\mathbb{P}(X)_Y \underset{\mathbb{S}_Y^{\beta}}{\overset{\mathbb{S}_Y^{\alpha}}{\rightrightarrows}} \mathbb{P}(X + X + Y)_Y$$

it clearly then has the right universal property of the double exponential. □

3.3 The upper and lower power monad axiom

In [9] an axiomatic account of the upper and lower power constructions is developed. Both are submonads of the double power monad. The defining characteristic of these power constructions, denoted by P_U and P_L respectively, is that their points (i.e. morphism $\mathcal{C}(Z, P_U(X))$, $\mathcal{C}(Z, P_L(X))$ respectively) are an order isomorphism with internal meet (respectively join) semilattice homomorphisms $\mathbb{S}^X \to \mathbb{S}^Z$. Notice that just as in the double power construction it is easy, by change of base, to check that $P_U^Y(X_Y) \cong P_U(X)_Y$ and $P_L^Y(X_Y) \cong P_L(X)_Y$

where the Y in P_L^Y indicates the lower power construction relative to $\mathcal{C}/Y$, and similarly for the upper power construction. Explicitly,

$$P_L(X) \overset{j_X^L}{\hookrightarrow} \mathbb{P}(X)$$

is constructed as the equalizer of

$$\mathbb{P}(X) \underset{(\mathbb{S}^{0_{\mathbb{S}^X}},g)}{\overset{(0_{\mathbb{S}}!^{\mathbb{P}(X)},f)}{\rightrightarrows}} \mathbb{S} \times \mathbb{P}(X+X)$$

where f is the exponential transpose of

$$\mathbb{S}^{\mathbb{S}^X} \times \mathbb{S}^X \times \mathbb{S}^X \to \mathbb{S}, \quad (\Lambda, a, b) \mapsto \Lambda(a \sqcup_{\mathbb{S}^X} b)$$

and g is the exponential transpose of

$$\mathbb{S}^{\mathbb{S}^X} \times \mathbb{S}^X \times \mathbb{S}^X \to \mathbb{S}, \quad (\Lambda, a, b) \mapsto \Lambda(a) \sqcup_{\mathbb{S}} \Lambda(b).$$

The notation (P_L, η^L, μ^L) is used for the induced monad and we use $\Diamond_X : \mathbb{S}^X \to \mathbb{S}^{P_L(X)}$ for the double exponential transpose of j_X^L. By construction, every internal join semilattice homomorphism $\alpha : \mathbb{S}^X \to \mathbb{S}^Y$ factors as $\mathbb{S}^{p_\alpha} \Diamond_X$ for some unique $p_\alpha : Y \to P_L(X)$ and this establishes an order isomorphism between $\sqcup$-SLat$[\mathbb{S}^X, \mathbb{S}^Y]$ and $\mathcal{C}(Y, P_L(X))$.

To develop a reasonable theory of these power constructions (for example to prove the Hofmann–Mislove theorem) it appears to be necessary to make the additional assumption that P_U is co-KZ and, order dually, that P_L is KZ. So, given the subject to hand, it makes sense to check that this additional assumption is also slice stable. We will look at the lower construction only; the upper construction is order dual.

Let us first gather a couple of consequences of Axiom 4.

Proposition 3.3.1. A. (Lower coverage theorem) *If*

$$E \overset{e}{\hookrightarrow} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

is an equalizer diagram in $\mathcal{C}$ then for any join semilattice homomorphism $\alpha : \mathbb{S}^X \to \mathbb{S}^Z$ such that

$$\alpha \sqcap_{\mathbb{S}^X} (\mathrm{Id} \times \mathbb{S}^f) = \alpha \sqcap_{\mathbb{S}^X} (\mathrm{Id} \times \mathbb{S}^g)$$

there exists a unique join semilattice homomorphism β such that $\beta \mathbb{S}^e = \alpha$.

B. *If $e : E \hookrightarrow X$ is a regular monomorphism then so is $P_L(e)$.*

Proof. A. Given such an α, there exists unique β such that $\beta\mathbb{S}^e = \alpha$; this is by Axiom 4. We just need to check that β is a join semilattice homomorphism. Certainly β preserves 0 since $\mathbb{S}^e$ and α do. To prove that β preserves binary join it is sufficient to prove that $\mathbb{S}^e \times \mathbb{S}^e$ is an epimorphism (in $\mathcal{C}_{\mathbb{P}}$) since both $\mathbb{S}^e$ and α preserve binary join. But $\mathbb{S}^e \times \mathbb{S}^e = (\mathbb{S}^e \times \mathbb{S}^{1_E})(\mathbb{S}^{1_X} \times \mathbb{S}^e)$ so it will follow that this morphism is an epimorphism provided we show that $1_X + e$ and $e + 1_E$ are both regular monomorphisms and then apply Axiom 4 twice. But by Axiom 2,

$$\begin{array}{ccc} E + E & \xrightarrow{[fe,fe]} & Y \\ {\scriptstyle e+1_E}\downarrow & & \downarrow{\scriptstyle \Delta_Y} \\ X + E & \xrightarrow{[(f,g),(fe,fe)]} & Y \times Y \end{array}$$

is a pullback diagram and regular monomorphisms (here Δ_Y) are pullback stable, so $e + 1_E$ is a regular monomorphism. $1_X + e$ is dealt with similarly.

B. This is an application of A and the defining property of the points of $P_L(E)$ and $P_L(X)$. $\square$

The order dual of Part A of the proposition is stated as an axiom in [10]. Part B can be applied to show that our final axiom is slice stable.

Axiom 8. The monad P_L is KZ and the monad P_U is co-KZ.

Proposition 3.3.2. *Given Axioms* 1–4 *and Axiom* 7*, Axiom* 8 *is slice stable.*

Proof. We look at the lower power monad only; the situation with the upper power monad is dual.

To see the proof consider the following commuting diagram constructed from X_f, an arbitrary object of $\mathcal{C}/Y$:

$$\begin{array}{ccccc} P_L^Y P_L^Y(X_f) & \xrightarrow{[\mu_L^Y]_{X_f}} & P_L^Y(X_f) & \xrightarrow{[P_L^Y \eta_L^Y]_{X_f}} & P_L^Y P_L^Y(X_f) \\ {\scriptstyle P_L^Y P_L^Y((\mathrm{Id},f))}\downarrow & & {\scriptstyle P_L^Y((\mathrm{Id},f))}\downarrow & & {\scriptstyle P_L^Y P_L^Y((\mathrm{Id},f))}\downarrow \\ P_L^Y P_L^Y(X_Y) & \xrightarrow{[\mu_L^Y]_{X_Y}} & P_L^Y(X_Y) & \xrightarrow{[P_L^Y \eta_L^Y]_{X_Y}} & P_L^Y P_L^Y(X_Y). \end{array}$$

The assertion that the monad defined by P_L is KZ is, by definition (e.g. [4, Lemma B1.1.12]), exactly the assertion that μ_L is right adjoint to $P_L\eta_L$ in the order enrichment. But since $\mu_L \circ P_L\eta_L = \mathrm{Id}$ this is equivalent to the assertion that $P_L\eta_L \circ \mu_L \sqsubseteq \mathrm{Id}$. This last assertion is sufficient to show that the bottom row of the diagram is less than or equal to $\mathrm{Id}_{P_L^Y P_L^Y(X_Y)}$ since we have commented already that $P_L^Y(X_Y) \cong P_L(X)_Y$. Therefore

$$[P_L^Y \eta_L^Y]_{X_f} \circ [\mu_L^Y]_{X_f} \sqsubseteq \mathrm{Id}_{P_L^Y P_L^Y(X_f)}$$

since $P_L^Y P_L^Y((\mathrm{Id}, f))$ is a regular monomorphism as (Id, f) is. □

The axiom given in [9] relevant to this is the stronger assertion that the Kleisli category $\mathcal{C}_{P_L}^{\mathrm{op}}$ is Cauchy complete. It is shown in that paper that an assumption of Cauchy completeness implies that the lower power monad is KZ. In application it is only the KZ property of the power monad that is exploited and so the result just shown is enough to prove that all the results of [9] are slice stable.

From now on we will assume that $\mathcal{C}$ is a category that satisfies all the axioms.

4 Properties of discrete objects in $\mathcal{C}$

As an application we now prove some results from locale theory. None of the results are new; the purpose of the exposition is to (i) show that they can all be shown axiomatically and (ii) provide some definitions and lemmas that are necessary for our final section on Joyal and Tierney's localic slice stability result. We only discuss the lower case that arises through our axiomatisation of the lower power monad. The upper case is exactly order dual and is not discussed.

The first subsection recalls the definition of an open map and develops some basic results about open maps. We recall how discrete objects can be defined using open maps and that the resulting full subcategory, $\mathsf{Dis}_{\mathcal{C}}$, is regular. The second subsection shows how the order dual of the Hofmann–Mislove theorem can be applied to show that discrete objects are exponentiable. The third subsection describes how the isomorphism inherent in this observation can be defined using relational composition. The final two subsections introduce the ideal completion of a preorder and develop its theory, essentially by applying our axiomatic relational composition to recover what is familiar set theoretically.

This section is long since the proofs are based on a categorical axiomatisation and so we no longer have certain basic set theoretic results available. The logically more sophisticated reader may wish to simply observe that the usual set theoretic arguments are all available using only the regular fragment of set theory and since the relevant category of discrete objects can be shown axiomatically to be regular all the arguments can be deployed within it. This makes the familiar lattice theoretic proofs available in the axiomatic context that is given and so this provides an alternative route for the proofs in this section. However we have chosen to ignore this approach and just prove the results directly from the axioms.

4.1 Open maps

To proceed we are going to need to recall some results about open maps relative to $\mathcal{C}$. Following the usual definition for the case when $\mathcal{C}$ is the category of locales, a morphism $f : X \to Y$ of $\mathcal{C}$ is said to be *open* if $\mathbb{S}^f$ has a left adjoint $\exists_f$ satisfying the Frobenius reciprocity condition

$$\sqcap_{\mathbb{S}^Y}(\exists_f \times \mathrm{Id}_{\mathbb{S}^Y}) = \exists_f \sqcap_{\mathbb{S}^X} (\mathrm{Id}_{\mathbb{S}^X} \times \mathbb{S}^f).$$

The reason for our interest in open maps is that they can be used to determine when an object is discrete:

Definition 4.1.1. An object X of $\mathcal{C}$ is said to be *discrete* provided the maps $!^X : X \to 1$ and $\Delta_X : X \hookrightarrow X \times X$ are open.

This captures the usual localic notion of discrete. We use $\mathsf{Dis}_{\mathcal{C}}$ to denote the full subcategory of discrete objects of $\mathcal{C}$.

It can be shown axiomatically that open maps are pullback stable, see [8]. Standard categorical arguments can be deployed to show that (i) any morphism between discrete objects is open and (ii) finite limits in $\mathsf{Dis}_{\mathcal{C}}$ are created in $\mathcal{C}$.

The paper [8] also shows that the pullback stability of open maps can be strengthened as follows: if

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{p^* f} & Z \\ {\scriptstyle f^* p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback diagram in $\mathcal{C}$ and p is open then $\exists_{f^* p}$ satisfies the Beck–Chevalley condition; that is $\exists_{f^* p}\mathbb{S}^{p^* f} = \mathbb{S}^f \exists_p$. Further $\exists_{f^* p}$ is unique in the sense that if $\alpha : \mathbb{S}^{X \times_Y Z} \to \mathbb{S}^X$ is any other join semilattice homomorphism satisfying Frobenius reciprocity on $f^* p$ and $\alpha \mathbb{S}^{p^* f} = \mathbb{S}^f \exists_p$ then $\alpha = \exists_{f^* p}$. As an application we show that in certain situations $\exists_{f^* p}$ can be calculated explicitly:

Lemma 4.1.2. *Given objects A and Z of $\mathcal{C}$, if $!^A : A \to 1$ is open then $\exists_{\pi_1} : \mathbb{S}^{Z \times A} \to \mathbb{S}^Z$ is given by*

$$[\exists_{\pi_1}]_W(a) = [\exists_A]_{W \times Z}(a)$$

for each object W of $\mathcal{C}$ and each $a : W \times Z \times A \to \mathbb{S}$. In other words, $\exists_{\pi_1} = \exists_A^Z$ where exponentiation is in $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$.

Of course the notation $\exists_A : \mathbb{S}^A \to \mathbb{S}$ is for the left adjoint to $\mathbb{S}^{!^A}$. We say that A is *open* if $!^A$ is an open map (though the term *overt* is also used in the literature). This technical result will be of use later when we are describing the exponentiability of discrete objects.

Proof. Define $\alpha : \mathbb{S}^{Z \times A} \to \mathbb{S}^Z$ by $\alpha_W(a) = [\exists_A]_{W \times Z}(a)$. It is routine to verify that this is natural in W and that α is a join semilattice homomorphism since $\exists_A$ is. But by the naturality of $\exists_A$ we have that $\alpha \mathbb{S}^{\pi_2} = \mathbb{S}^{!^Z} \exists_A$ and since α satisfies Frobenius reciprocity on $\pi_1 : Z \times X \to Z$ (as $\exists_A$ satisfies Frobenius reciprocity on $!^A : A \to 1$) we have that $\alpha = \exists_{\pi_1}$ by the uniqueness of maps satisfying Beck–Chevalley on the pullback square:

$$\begin{array}{ccc} Z \times A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle !^A} \\ Z & \xrightarrow{!^Z} & 1 \end{array}$$

This finishes the proof. □

It can also be shown (see [8]) that a subobject is open if and only if it is an open regular monomorphism. In one direction this is shown by observing that the map $\ulcorner \sqcap \urcorner : \mathbb{S} \to \mathbb{S}^{\mathbb{S}}$, given by the exponential transpose in $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ of the meet operation $\sqcap : \mathbb{S} \times \mathbb{S} \to \mathbb{S}$, is a left adjoint to $\mathbb{S}^{\mathbb{S}} \overset{\mathbb{S}^{1_{\mathbb{S}}}}{\to} \mathbb{S}$. This left adjoint, it can be shown, witnesses that $1_{\mathbb{S}}$ is an open map and since $1_{\mathbb{S}}$ is a regular monomorphism (its domain is the terminal object) we have that every open subobject is an open regular monomorphism. In the other direction if $i : X_0 \hookrightarrow X$ is a regular monomorphism and i is an open map then it can be verified that i is the pullback of $1_{\mathbb{S}}$ along the open $\exists_i(1_{\mathbb{S}^{X_0}})$ (i.e. $[\exists_i]_1(X_0 \overset{1_{\mathbb{S}}!^{X_0}}{\to} \mathbb{S})$). It follows that for any open subobject $i : X_0 \hookrightarrow X$ we have that

$$\exists_i \mathbb{S}^i = a \sqcap (\_),$$

where $a = \exists_i(1_{\mathbb{S}^{X_0}})$.

Our final observation about $\mathsf{Dis}_{\mathcal{C}}$, which we again recall from [8], is that it forms a regular category and so has image factorizations. If $f : X \to Y$ is a map between discrete objects then it is open and its image is the open subobject $\exists_f(1_{\mathbb{S}^X})$ of Y. Note that this also implies that monomorphisms are regular in $\mathsf{Dis}_{\mathcal{C}}$ since the image factorization of a monomorphism is itself and open subobjects are regular monomorphisms.

Notation 4.1.3. We need to make a remark on notation. For any natural transformation $\alpha : \mathbb{S}^X \to \mathbb{S}^Y$ if $a : 1 \to \mathbb{S}^X$ then of course $\alpha(a) : 1 \to \mathbb{S}^Y$ is also, by definition, a natural transformation. However, by Yoneda's lemma, it is determined uniquely by the open $[\alpha]_1 \ulcorner a \urcorner$ where $\ulcorner a \urcorner : X \to \mathbb{S}$ is the exponential transpose of a. In the following we will not use notation to distinguish between (i) the morphism $\ulcorner a \urcorner : X \to \mathbb{S}$, (ii) the natural transformation $a : 1 \to \mathbb{S}^X$ or (iii) the open subobject $(\ulcorner a \urcorner)^* 1 \hookrightarrow X$ referring to each ((i), (ii) or (iii)) generically as 'an open of X' and always using the notation a.

On the other hand, for situations where we have $R : X \times Y \to \mathbb{S}$ we will use $\ulcorner R \urcorner$ for the exponential transpose $X \to \mathbb{S}^Y$. Although this introduces a notational difference between the binary and nullary situations, it does seem to strike the right balance by not overburdening the notation whilst at the same time retaining clarity when multiple variables need to be introduced.

Putting this remark on notation to use, and as another simple application of the Beck–Chevalley condition on pullback diagrams of open maps, we show that if a discrete object has a point then the singleton subset corresponding to that point is inhabited:

Lemma 4.1.4. *If A is a discrete object of $\mathcal{C}$ and there exists $p : 1 \to A$ then*

$$\exists_A(\ulcorner \Delta_A \urcorner p) = 1_{\mathbb{S}}.$$

Proof. The open $\ulcorner \Delta_A \urcorner p$, i.e. $1 \overset{p}{\to} A \overset{\ulcorner \Delta_A \urcorner}{\longrightarrow} \mathbb{S}^A$, is equal to $A \overset{(\mathrm{Id}_A, p!^A)}{\longrightarrow} A \times A \overset{\Delta_A}{\to} \mathbb{S}$ so following the notational convention just remarked upon,

$$\ulcorner \Delta_A \urcorner p = \Delta_A(\mathrm{Id}_A, p!^A) = \mathbb{S}^{(\mathrm{Id}_A, p!^A)} \Delta_A.$$

By Beck–Chevalley applied to the pullback square

$$\begin{array}{ccc} A & \xrightarrow{(\mathrm{Id}_A, p!^A)} & A \times A \\ {\scriptstyle !^A}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ 1 & \xrightarrow{p} & A \end{array}$$

we have that

$$\exists_A(\ulcorner \Delta_A \urcorner p) = \exists_A \mathbb{S}^{(\mathrm{Id}_A, p!^A)} \Delta_A = \mathbb{S}^p \exists_{\pi_2} \Delta_A = \mathbb{S}^p 1_{\mathbb{S}^A} = 1_{\mathbb{S}},$$

where $\exists_{\pi_2} \Delta_A = 1_{\mathbb{S}^A}$ since $\Delta_A = \exists_{\Delta_A}(1_{\mathbb{S}^A})$ and $\exists_{\pi_2} \exists_{\Delta_A} = \mathrm{Id}_{\mathbb{S}^A}$ by uniqueness of left adjoints since $\exists_{\pi_2} \exists_{\Delta_A}$ is left adjoint to $\mathbb{S}^{\Delta_A} \mathbb{S}^{\pi_2} = \mathbb{S}^{\mathrm{Id}_A} = \mathrm{Id}_{\mathbb{S}^A}$. □

The following lemma will be required later. Intuitively, it is showing that any inhabited subset of 1 must be the whole of 1:

Lemma 4.1.5. *If* $i : I_0 \hookrightarrow 1$ *is an open subobject of* 1 *such that* $\exists_i(1) = 1$ *then* i *is an isomorphism.*

Proof. By definition, i is the pullback of $1 \overset{1_{\mathbb{S}}}{\to} \mathbb{S}$ along some $a_{I_0} : 1 \to \mathbb{S}$. Beck–Chevalley applied to the pullback square implies that $\exists_i \mathbb{S}^{!^{I_0}} = \mathbb{S}^{a_{I_0}} \ulcorner \sqcap \urcorner$ where $\ulcorner \sqcap \urcorner$ is the exponential transpose of $\sqcap : \mathbb{S} \times \mathbb{S} \to \mathbb{S}$. It follows that $\mathbb{S}^{a_{I_0}} \ulcorner \sqcap \urcorner(1) = 1$ since $\mathbb{S}^{!^{I_0}}$ preserves 1 and $\exists_i$ preserves 1 by assumption. It follows that $a_{I_0} = 1_{\mathbb{S}}$ since $\ulcorner \sqcap \urcorner(1)$ is the identity on $\mathbb{S}$ and so i is isomorphic to the identity on 1. □

4.2 Application of the Hofmann–Mislove theorem to exponentiability

In this subsection we recall the main result of [9] which is an axiomatic proof of the existence of an order isomorphism,

$$\mathcal{C}(1, P_L(X)) \cong \{X_0 \hookrightarrow X \mid X_0 \text{ open}, X_0 \hookrightarrow X \text{ weakly closed}\}. \qquad (*)$$

This result can be seen to be the order dual (in the order enrichment of $\mathcal{C}$) of the well-known Hofmann–Mislove theorem, see [2]. It is equivalent to Bunge–Funk's constructive description of the points of the lower power locale, see [1]. The reason for recalling it is that it, together with a slice stable account of $\mathcal{C}$, allows us to prove axiomatically the well-known result (made explicit by Vickers in [13]) that $\mathbb{S}^A \cong P_L(A)$ for discrete objects A. It is the application of this isomorphism that will guide much of the technical work for the rest of the paper so the final parts of this subsection are devoted to giving an explicit description of how the isomorphism works.

Of course, we must now give the definition of weakly closed:

Definition 4.2.1. A monomorphism $X_0 \hookrightarrow X$ is a *weakly closed subobject* if it exists as a lax equalizer of a diagram $f, g : X \rightrightarrows Y$ universally setting $g \sqsubseteq f$ where f factors via the terminal object.

This recovers the usual localic definition in the case where $\mathcal{C}$ is the category of locales.

We must recall how the isomorphism $(*)$ works: it sends a point $p_\alpha : 1 \to P_L(X)$ to the lax equalizer universally setting $\eta^L_X : X \to P_L(X)$ less than or equal to

$$X \xrightarrow{!^X} 1 \xrightarrow{p_\alpha} P_L(X).$$

In the other direction, $p_{\exists_{X_0}\mathbb{S}^i}$ is the point of $P_L(X)$ corresponding to weakly closed $X_0 \overset{i}{\hookrightarrow} X$ with X_0 open.

What happens when X is discrete? We have the following observation which is an axiomatic account of a result in [13]:

Lemma 4.2.2. *If A is a discrete object then for any regular monomorphism $i : X_0 \hookrightarrow A$ the following are equivalent:*

(a) *X_0 is an open object of $\mathcal{C}$ and i is a weakly closed subobject of A.*

(b) *i is an open subobject.*

Proof. (a) $\Rightarrow$ (b). The diagonal $\Delta_{X_0} : X_0 \hookrightarrow X_0 \times X_0$ is open since it is the pullback (along $i \times i$) of the open diagonal $\Delta_A : A \hookrightarrow A \times A$. Therefore X_0 is discrete and since any morphism between discrete objects is open we have that i is open.

(b) $\Rightarrow$ (a). Certainly X_0 is open since the unique map $!^{X_0}$ factors as

$$X_0 \overset{i}{\hookrightarrow} A \overset{!^A}{\longrightarrow} 1$$

and both factors are open. Next we show that i is weakly closed. By the definition of being an open subobject, there exists a morphism $a : A \to \mathbb{S}$ such that $i : a \hookrightarrow A$ is the pullback of $1 : 1 \to \mathbb{S}$ along a. We show that $a \hookrightarrow A$ is the lax equalizer universally setting η^L_A less than or equal to $p_\alpha !^A$ where α is $\mathbb{S}^A \overset{\mathbb{S}^i}{\to} \mathbb{S}^{X_0} \overset{\exists_{X_0}}{\to} \mathbb{S}$; equivalently, $\alpha = \exists_A(a \sqcap (\_))$. Note firstly that $\eta^L_A i \sqsubseteq p_\alpha !^A i$ since $\mathbb{S}^i \mathbb{S}^{!^A} = \mathbb{S}^{!^{X_0}}$ and $\mathrm{Id} \sqsubseteq \mathbb{S}^{!^{X_0}} \exists_{X_0}$ so to complete it is sufficient to check that if there is any $E \overset{e}{\to} A$ with the property that $\eta^L_A e \sqsubseteq p_\alpha !^A e$ then e must factor through i. This will show that i is the lax equalizer universally setting η^L_A less than or equal to $p_\alpha !^A$.

By change of base (to stage E) we can assume that we are given $1 \overset{e}{\to} A$ with the property that $\eta^L_A e \sqsubseteq p_\alpha !^A e$. By taking exponential transpose this implies that $\mathbb{S}^A \overset{\mathbb{S}^e}{\to} \mathbb{S}$ is less than or equal to α in the order enrichment. Say the subobject $1 \overset{e}{\to} A$ (which must be open since 1 is discrete) is classified by $a_e : A \to \mathbb{S}$, i.e. $a_e = \exists_e(1)$. Therefore since $1 = \mathbb{S}^e(a_e)$ we have that $1 = \alpha(a_e) = \exists_A(a \sqcap a_e)$. Now consider the pullback square

$$\begin{array}{ccc} I_0 & \overset{p}{\longrightarrow} & X_0 \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle i} \\ 1 & \underset{e}{\longrightarrow} & A \end{array} \qquad (+)$$

Since i_0 factors as $!^A i p$, we must have that $\exists_{I_0} = \exists_A \exists_i \exists_p$ (as the left adjoint to $\mathbb{S}^{i_o} = \mathbb{S}^p \mathbb{S}^i \mathbb{S}^{!^A}$ is unique and certainly p is open as it is the pullback of the open e). But then

$$\begin{aligned}\exists_{I_0}(1) = \exists_A \exists_i \exists_p(1) &= \exists_A \exists_i \exists_p(\mathbb{S}^{i_o} 1)\\ &= \exists_A \exists_i \mathbb{S}^i \exists_e(1) \qquad \text{(x)}\\ &= \exists_A(a \sqcap a_e) = 1,\end{aligned}$$

where step (x) is by Beck–Chevalley of the pullback square (+). But then, by Lemma 4.1.5, we have that i_0 is an isomorphism; and this is sufficient to show that e factors through i as required. □

For discrete A therefore we have an order isomorphism between open subobjects of A and join semilattice homomorphisms $\mathbb{S}^A \to \mathbb{S}$. The next lemma clarifies how this order isomorphism works; in other words, it describes how the order isomorphism $(*)$, recalled above, specialises when X is restricted to being a discrete object.

Lemma 4.2.3. *If A is a discrete object of $\mathcal{C}$ then*

$$\mathcal{C}(A, \mathbb{S}) \to \sqcup\text{-SLat}[\mathbb{S}^A, \mathbb{S}], \quad a \mapsto \exists_A(a \sqcap (\_))$$

is an order isomorphism whose inverse sends any $\alpha : \mathbb{S}^A \to \mathbb{S}$ to $\alpha \ulcorner \Delta_A \urcorner$ where $\ulcorner \Delta_A \urcorner : A \to \mathbb{S}^A$ is the exponential transpose of the diagonal.

Proof. The proof is essentially an application of our description of the order isomorphism $(*)$ which we have recalled above. If $a \overset{i}{\hookrightarrow} A$ is open then the corresponding join semilattice homomorphism under $(*)$ is the map $\exists_a \mathbb{S}^i$; but $\exists_a$ factors as $\exists_A \exists_i$ (by the uniqueness of left adjoints since $!^a$ factors as $!^A i$) and $\exists_i \mathbb{S}^i = a \sqcap (\_)$; so the join semilattice homomorphism corresponding to a is $\exists_A(a \sqcap (\_))$.

In the other direction, given a join semilattice homomorphism $\alpha : \mathbb{S}^A \to \mathbb{S}$, we have established in the previous lemma that the corresponding weakly closed map, $i : X_0 \hookrightarrow A$ say, is open. To complete this proof we must show that $\exists_i(1) = \alpha \ulcorner \Delta_A \urcorner$. Firstly, note that by our explicit description of $(*)$ recalled above

$$\alpha = \exists_{X_0} \mathbb{S}^i.$$

But, further, i factors as $X_0 \overset{(i, \mathrm{Id}_{X_0})}{\hookrightarrow} A \times X_0 \overset{\pi_2}{\to} A$ where π_2 is open since it is the pullback of the open $!^{X_0}$. The proof therefore reduces to showing

$$\exists_{\pi_2} \exists_{(i, \mathrm{Id}_{X_0})}(1) = \exists_{X_0} \mathbb{S}^i \ulcorner \Delta_A \urcorner.$$

Now by Beck–Chevalley applied to the pullback square

$$\begin{array}{ccc} X_0 & \xrightarrow{i} & A \\ \downarrow{\scriptstyle (i,\mathrm{Id}_{X_0})} & & \downarrow{\scriptstyle \Delta_A} \\ A \times X_0 & \xrightarrow{\mathrm{Id}_A \times i} & A \times A \end{array}$$

(and using the fact that $\mathbb{S}^i(1) = 1$) we are in fact reduced to checking that

$$\exists_{\pi_2} \mathbb{S}^{\mathrm{Id}_A \times i}(\exists_{\Delta_A}(1)) = \exists_{X_0} \mathbb{S}^i \ulcorner \Delta_A \urcorner.$$

However this is immediate by taking exponential transpose since we know that (Lemma 4.1.2) $\exists_{\pi_2} = \exists^A_{X_0}$ because X_0 is open. □

Proposition 4.2.4. *If A is discrete then the presheaf $\mathbb{S}^A$ is representable and is naturally isomorphic to $P_L(A)$.*

Proof. The proposition is equivalent to the assertion that the exponential $\mathbb{S}^A$ exists in $\mathcal{C}$ and is isomorphic to $P_L(A)$.

But for any object X of $\mathcal{C}$ we have

$$\begin{aligned} \mathcal{C}(X \times A, \mathbb{S}) &\cong \mathcal{C}/X(A_X, \mathbb{S}_X) \\ &\cong \{I \hookrightarrow A_X \text{ open relative to } X\} \\ &\cong \mathcal{C}/X(1, P^X_L(A_X)) \\ &\cong \mathcal{C}(X, P_L(A)). \end{aligned}$$

The second to last line uses the previous lemmas carried out in the slice $\mathcal{C}/X$. The last line is because the lower power monad is slice stable. □

There is therefore an isomorphism in $\mathcal{C}$ from $P_L(A)$ to $\mathbb{S}^A$ for discrete objects A. In later sections we will need the following explicit description of this isomorphism.

Lemma 4.2.5. (a) *The natural order isomorphism $\mathcal{C}(X, P_L(A)) \to \mathcal{C}(X, \mathbb{S}^A)$ established in the previous proposition sends any $p_\alpha : X \to P_L(A)$ to the double exponential transpose of*

$$A \xrightarrow{\ulcorner \Delta_A \urcorner} \mathbb{S}^A \xrightarrow{\alpha} \mathbb{S}^X.$$

(b) *The isomorphism* $\phi_A : P_L(A) \to \mathbb{S}^A$ *established in the previous proposition is given by the double exponential transpose of*

$$A \xrightarrow{\ulcorner \Delta_A \urcorner} \mathbb{S}^A \xrightarrow{\Diamond_A} \mathbb{S}^{P_L(A)}.$$

Proof. Firstly, (b) is immediate from (a) since $\phi_A : P_L(A) \to \mathbb{S}^A$ is the mate of the identity on $P_L(A)$ under $\mathcal{C}(X, P_L(A)) \cong \mathcal{C}(X, \mathbb{S}^A)$ and the identity on $P_L(A)$ is equal to $p_{\Diamond_A}$.

For (a) this is just from the previous lemma by slice stability. □

4.3 Relational composition

In this subsection we describe the order isomorphism just established in terms of axiomatic relational composition. It seems that the most appropriate definition of relational composition is found by not restricting to discrete objects but by only making the minimal (and surely necessary) assumption that the object that is both codomain and domain in the definition of composition is open. Although the operation is then only partially defined it has all the right properties.

Definition 4.3.1. If A is an open object of $\mathcal{C}$ and R_1 an open of $W \times A$ and R_2 an open of $A \times X$ then define $R_1; R_2$, an open of $W \times X$, by

$$R_1; R_2 = \exists_{\pi_{13}}(\mathbb{S}^{\pi_{12}} R_1 \sqcap \mathbb{S}^{\pi_{23}} R_2).$$

Our first observation is that this definition makes sense:

$$\exists_{\pi_{13}} : \mathbb{S}^{W \times A \times X} \to \mathbb{S}^{W \times X}$$

exists since $!^A : A \to 1$ is an open map and therefore so too is $\pi_{13} : W \times A \times X \to W \times X$ as it is the pullback of $!^A$ along $!^{W \times X}$.

Our second observation is that if W, A and X are discrete then this definition recovers relational composition. This is because $\exists_{\pi_{13}}$ defines image factorization in the category of discrete objects relative to $\mathcal{C}$.

Finally, we must of course check that:

Lemma 4.3.2. *Where* ; *is defined it is associative.*

Proof. Say R_1 is an open of $W \times A$, R_2 is an open of $A \times B$ and R_3 is an open of $B \times Y$ where A and B are open objects of $\mathcal{C}$. We must check that

$$(R_1; R_2); R_3 = R_1; (R_2; R_3). \qquad (*)$$

But the LHS of $(*)$ is

$$\begin{aligned}
&\exists_{\pi_{13}}(\mathbb{S}^{\pi_{12}}\exists_{\pi_{13}}(\mathbb{S}^{\pi_{12}}R_1 \sqcap \mathbb{S}^{\pi_{23}}R_2) \sqcap \mathbb{S}^{\pi_{23}}R_3)\\
&\quad = \exists_{\pi_{13}}(\exists_{\pi_{134}}\mathbb{S}^{\pi_{123}}(\mathbb{S}^{\pi_{12}}R_1 \sqcap \mathbb{S}^{\pi_{23}}R_2) \sqcap \mathbb{S}^{\pi_{23}}R_3)\\
&\quad = \exists_{\pi_{13}}\exists_{\pi_{134}}((\mathbb{S}^{\pi_{12}}R_1 \sqcap \mathbb{S}^{\pi_{23}}R_2) \sqcap \mathbb{S}^{\pi_{134}}\mathbb{S}^{\pi_{23}}R_3)\\
&\quad = \exists_{\pi_{14}}(\mathbb{S}^{\pi_{12}}R_1 \sqcap \mathbb{S}^{\pi_{23}}R_2 \sqcap \mathbb{S}^{\pi_{34}}R_3),
\end{aligned}$$

where the first line is by Beck–Chevalley for the pullback square

$$\begin{array}{ccc}
W \times A \times B \times Y & \xrightarrow{\pi_{123}} & W \times A \times B\\
\downarrow{\scriptstyle \pi_{134}} & & \downarrow{\scriptstyle \pi_{13}}\\
W \times B \times Y & \xrightarrow{\pi_{12}} & W \times B
\end{array}$$

The second line exploits the fact that $\pi_{134} : W \times A \times B \times Y \to W \times B \times Y$ is open (it is the pullback of the open π_{13}) and so satisfies Frobenius reciprocity. The last line is because $\pi_{14} : W \times A \times B \times Y \to W \times Y$ factors as

$$W \times A \times B \times Y \xrightarrow{\pi_{134}} W \times B \times Y \xrightarrow{\pi_{13}} W \times Y;$$

therefore $\mathbb{S}^{\pi_{134}}\mathbb{S}^{\pi_{13}} = \mathbb{S}^{\pi_{14}}$ and so $\exists_{\pi_{13}}\exists_{\pi_{134}} = \exists_{\pi_{14}}$ by uniqueness of left adjoints.

Symmetrically, the RHS of $(*)$ can be reduced to $\exists_{\pi_{14}}(\mathbb{S}^{\pi_{12}}R_1 \sqcap \mathbb{S}^{\pi_{23}}R_2 \sqcap \mathbb{S}^{\pi_{34}}R_3)$ and so the proof is complete. □

Beyond helping with spatial intuitions our relational composition operation is also key as it allows us to be more explicit about the order isomorphism $\mathcal{C}(X \times A, \mathbb{S}) \cong \sqcup\text{-SLat}(\mathbb{S}^A, \mathbb{S}^X)$ established above.

Lemma 4.3.3. *If A is a discrete object of $\mathcal{C}$ then the map*

$$\mathcal{C}(A \times X, \mathbb{S}) \to \sqcup\text{-SLat}[\mathbb{S}^A, \mathbb{S}^X], \quad R \mapsto \alpha_R$$

with α_R defined by $[\alpha_R]_W(I) = I; R$, is an order isomorphism whose inverse sends any $\alpha : \mathbb{S}^A \to \mathbb{S}^X$ to the exponential transpose of $\alpha\ulcorner\Delta_A\urcorner : A \to \mathbb{S}^X$.

In other words, for every open relation R on $A \times X$, $\ulcorner R\urcorner : A \to \mathbb{S}^X$ factors as $\alpha_R\ulcorner\Delta_A\urcorner$ where α_R is defined in terms of relational composition.

Proof. Because of (a) of Lemma 4.2.5 we must but show that the image of R under the isomorphism of that lemma is indeed α_R. It is sufficient to check that

$$\ulcorner R\urcorner = \alpha_R\ulcorner\Delta_A\urcorner$$

which is the same as checking that $\Delta_R; R = R$. This is true by change of base since Δ_{A_X} is the identity for relational composition in $\mathsf{Dis}_{\mathcal{C}/X}$. Alternatively, we can use Beck–Chevalley again:

$$\begin{aligned}
\Delta_R; R &= \exists_{\pi_{13}}(\mathbb{S}^{\pi_{12}}\Delta_A \sqcap \mathbb{S}^{\pi_{23}}R) \\
&= \exists_{\pi_{13}}(\mathbb{S}^{\pi_{12}}\exists_{\Delta_A}(1_{\mathbb{S}^A}) \sqcap \mathbb{S}^{\pi_{23}}R) \\
&= \exists_{\pi_{13}}(\exists_{\Delta_A\times \mathrm{Id}_X}\mathbb{S}^{\pi_1}(1_{\mathbb{S}^A}) \sqcap \mathbb{S}^{\pi_{23}}R) && \text{(a)} \\
&= \exists_{\pi_{13}}\exists_{\Delta_A\times \mathrm{Id}_X}(\mathbb{S}^{\pi_1}(1_{\mathbb{S}^A}) \sqcap \mathbb{S}^{\Delta_A\times \mathrm{Id}_X}\mathbb{S}^{\pi_{23}}R) && \text{(b)} \\
&= \exists_{\mathrm{Id}_{A\times X}}(1_{\mathbb{S}^{A\times X}} \sqcap R) = R,
\end{aligned}$$

where (a) is by Beck–Chevalley on the pullback square

$$\begin{array}{ccc}
A \times X & \xrightarrow{\pi_1} & A \\
{\scriptstyle \Delta_A\times \mathrm{Id}_X}\downarrow & & \downarrow{\scriptstyle \Delta_A} \\
A \times A \times X & \xrightarrow{\pi_{12}} & A \times A
\end{array}$$

and (b) is because $\Delta_A \times \mathrm{Id}_X$ is open since it is the pullback of the open map Δ_A. □

Corollary 4.3.4. *Where it is defined relational composition is mapped to morphism composition under the isomorphism of the lemma.*

Proof. This is immediate since the lemma describes any join semilattice corresponding to a relation in terms of relational composition and we have already checked that relational composition is associative. □

The following result makes use of this explicit description of the order isomorphism in terms of relational composition. It proves, for discrete objects at least, that product in $\mathcal{C}$ is given by a join semilattice tensor in $\mathcal{C}_{P_L}^{\mathrm{op}}$. In the case that $\mathcal{C} = \mathsf{Loc}$ this is equivalent to the basic observation that locale product is given by suplattice tensor.

Proposition 4.3.5. *If A and B are two discrete objects of $\mathcal{C}$ then,*

(i) *the map $\otimes : \mathbb{S}^A \times \mathbb{S}^B \to \mathbb{S}^{A\times B}$ defined by $\otimes = \sqcap_{\mathbb{S}^{A\times B}}(\mathbb{S}^{\pi_1} \times \mathbb{S}^{\pi_2})$ is universally join bilinear,*

(ii) *the map $\overline{\Diamond_{A\times B}\otimes} : P_L(A\times B) \to \mathbb{P}(A+B)$ defined as the double exponential transpose of $\mathbb{S}^A \times \mathbb{S}^B \xrightarrow{\otimes} \mathbb{S}^{A\times B} \xrightarrow{\Diamond_{A\times B}} \mathbb{S}^{P_L(A\times B)}$ is a monomorphism.*

Proof. (i) We must check that for any join semilattice homomorphism $\mathbb{S}^{A\times B} \to \mathbb{S}^Y$ that precomposition with $\otimes$ induces an order isomorphism between $\sqcup$-SLat$(\mathbb{S}^{A\times B}, \mathbb{S}^Y)$ and

$$\sqcup\text{-Bilinear}(\mathbb{S}^A \times \mathbb{S}^B, \mathbb{S}^Y).$$

By change of base we can assume that $Y = 1$. But we have the following series of order isomorphisms:

$$\begin{aligned}\sqcup\text{-Bilinear}(\mathbb{S}^A \times \mathbb{S}^B, \mathbb{S}) &\cong \sqcup\text{-SLat}(\mathbb{S}^A, P_L(B))\\ &\cong \sqcup\text{-SLat}(\mathbb{S}^A, \mathbb{S}^B)\\ &\cong \mathcal{C}(A\times B, \mathbb{S})\\ &\cong \sqcup\text{-SLat}(\mathbb{S}^{A\times B}, \mathbb{S}),\end{aligned}$$

where the first line is by exponential transpose and the definition of$P_L(B)$ as a subobject of $\mathbb{P}(B)$. Therefore we have but to check that in the reverse direction (i.e. from $\sqcup$-SLat$(\mathbb{S}^{A\times B}, \mathbb{S})$ to $\sqcup$-Bilinear$(\mathbb{S}^A \times \mathbb{S}^B, \mathbb{S})$) any join semilattice $\beta : \mathbb{S}^{A\times B} \to \mathbb{S}$ is mapped to $\beta\otimes$. From above such a β is given by $\exists_{A\times B}(R_\beta \sqcap (\_))$ for some unique open R_β of $A \times B$; the image of β in $\sqcup$-SLat$(\mathbb{S}^A, \mathbb{S}^B)$ is therefore α_{R_β}. On the other hand, the image of $\beta\otimes$ in $\sqcup$-SLat$(\mathbb{S}^A, \mathbb{S}^B)$ under the above order isomorphism is the composite

$$\mathbb{S}^A \xrightarrow{\ulcorner\beta\otimes\urcorner} P_L(B) \xrightarrow{\phi_B} \mathbb{S}^B.$$

Our proof of part (i) will then be complete provided that we can prove that these two join semilattice homomorphisms from $\mathbb{S}^A$ to $\mathbb{S}^B$ are equal. This can be achieved by checking for any $Z \xrightarrow{z} \mathbb{S}^A$ we have $\alpha_{R_\beta} z = \phi_B \ulcorner\beta\otimes\urcorner z$; but in fact we can assume $Z = 1$ (and so $z = I$ for some open I of A) by applying a change of base argument. Now $\ulcorner\beta\otimes\urcorner(I)$ is the point of $P_L(B)$ corresponding to the map $\mathbb{S}^B \xrightarrow{\beta(I\otimes\_)} \mathbb{S}$, i.e. to

$$\mathbb{S}^B \xrightarrow{\exists_{A\times B}(R_\beta \sqcap I\otimes(\_))} \mathbb{S}.$$

Its image under the isomorphism $P_L(B) \to \mathbb{S}^B$ is the open $\exists_{A\times B}(R_\beta \sqcap I \otimes (\ulcorner\Delta_B\urcorner))$ by Lemma 4.2.5, which we can see is equal to

$$\exists_B \exists_{\pi_2}(R_\beta \sqcap \mathbb{S}^{\pi_1} I \sqcap \mathbb{S}^{\pi_2}(\ulcorner\Delta_B\urcorner))$$

by using the definition of $\otimes$ and the fact that $\exists_{A\times B} = \exists_B \exists_{\pi_2}$ since $!^{A\times B} =!^B \pi_2$ for $\pi_2 : A \times B \to B$. Since π_2 is open (as it is the pullback of the open map $!^A$)

we have that this is equal to

$$\exists_B[\exists_{\pi_2}(R_\beta \sqcap \mathbb{S}^{\pi_1} I) \sqcap (\ulcorner \Delta_B \urcorner)] = \exists_{\pi_2}(R_\beta \sqcap \mathbb{S}^{\pi_1} I).$$

But we are then done since $\alpha_{R_\beta} I$ is the open $I; R_\beta = \exists_{\pi_2}(R_\beta \sqcap \mathbb{S}^{\pi_1} I)$.

(ii) is immediate given the universal property of $\otimes$ just established. Note that the double exponential transpose of $\mathbb{S}^{A\times B} \xrightarrow{\Diamond_{A\times B}} \mathbb{S}^{P_L(A\times B)}$ is, by definition, the regular monomorphism $j^L_{A\times B} : P_L(A \times B) \hookrightarrow \mathbb{P}(A \times B)$. □

An appendix has been included examining the additional assumption (which is made in [10]) that $\otimes : \mathbb{S}^X \times \mathbb{S}^Y \to \mathbb{S}^{X\times Y}$ is universal join bilinear for all objects X and Y. The appendix shows that this additional assumption is slice stable in the presence of the other axioms. It also includes a conjecture concerning the axiomatic account of localic subgroupoids.

4.4 The ideal completion of a preorder

In this subsection we introduce the ideal completion of a preorder and develop some basic properties of it. The main result is that this ideal completion construction is a specialisation of the construction given in [9].

Definition 4.4.1. A preorder relative to $\mathcal{C}$ is a discrete object B together with an open relation $\preceq_B \hookrightarrow B \times B$ which satisfies

(i) $\Delta_B \sqsubseteq \preceq_B$, and

(ii) $\preceq_B ; \preceq_B \sqsubseteq \preceq_B$.

Lemma 4.4.2. *The data for a preorder is equivalently a pair* $(B, \alpha : \mathbb{S}^B \to \mathbb{S}^B)$ *with the properties that* (i) B *is discrete,* (ii) α *is a join semilattice homomorphism* (iii) $\mathrm{Id}_{\mathbb{S}^B} \sqsubseteq \alpha$ *and* (iv) α *is idempotent.*

Proof. This is immediate from our observations about the order isomorphism between open subobjects of $B \times B$ and join semilattice homomorphisms $\mathbb{S}^B \to \mathbb{S}^B$; in particular the observation that relational composition maps to morphism composition. □

If B is a preorder then we use the notation $R_B \hookrightarrow B \times B \times B$ (and $B \times B \times B \xrightarrow{R_B} \mathbb{S}$) to denote the open subobject $\pi^*_{12}(\preceq_B) \wedge \pi^*_{13}(\preceq_B)$ (equivalently, the open $\mathbb{S}^{\pi_{12}}(\preceq_B) \sqcap \mathbb{S}^{\pi_{13}}(\preceq_B)$). Set theoretically we are of course thinking

$$R_B = \{(b_1, b_2, b_3) \mid b_1 \succeq_B b_2 \text{ and } b_1 \succeq_B b_3\}.$$

Definition 4.4.3. If B is a preorder relative to $\mathcal{C}$ then define $\mathrm{Idl}(B) \overset{i_B}{\hookrightarrow} \mathbb{S}^B$ to be the intersection (= pullback) of the equalizer

$$A^{\sqcap} \hookrightarrow \mathbb{S}^B \underset{\otimes\Delta_{\mathbb{S}^B}}{\overset{\alpha_{R_B}}{\rightrightarrows}} \mathbb{S}^{B\times B} \quad \text{and} \quad A^1 \hookrightarrow \mathbb{S}^B \underset{1_{\mathbb{S}}!^{\mathbb{S}^B}}{\overset{\exists_B}{\rightrightarrows}} \mathbb{S}.$$

Capturing the usual spatial notion we have that, by definition, an open I of a preorder B is said to be *an ideal* provided that

(i) $\exists_B(I) = 1_{\mathbb{S}}$ ("I is non-empty"),

(ii) $I; \succeq_B = I$ ("I is lower closed"), and

(iii) $I \otimes I \sqsubseteq I; R_B$ ("I is directed").

Note that condition (ii) is equivalent to $I; \succeq_B \sqsubseteq I$ since $\Delta_B \sqsubseteq \preceq_B$ by definition of preorder. The next proposition justifies our notation $\mathrm{Idl}(B)$ since it shows that the points of $\mathrm{Idl}(B)$ are in order isomorphism with the ideals of B.

Proposition 4.4.4. *For any open I of a preorder B, $I : 1 \to \mathbb{S}^B$ factors via $\mathrm{Idl}(B) \overset{i_B}{\hookrightarrow} \mathbb{S}^B$ if and only if I is an ideal of B.*

Proof. This is essentially by the construction of $\mathrm{Idl}(B)$. The only missing step is a proof that I is lower closed (equivalently $I; \succeq_B \sqsubseteq I$) if and only if $I; R_B \sqsubseteq I \otimes I$.

Firstly, say that $I; R_B \sqsubseteq I \otimes I$, then by application of $\mathbb{S}^{\Delta_B}$ we have that

$$\mathbb{S}^{\Delta_B} I; R_B \sqsubseteq I$$

since $I \otimes I = \mathbb{S}^{\pi_1} I \sqcap \mathbb{S}^{\pi_2} I$ (and certainly $\mathbb{S}^{\Delta_B}\mathbb{S}^{\pi_1} = \mathbb{S}^{\Delta_B}\mathbb{S}^{\pi_2} = \mathrm{Id}_{\mathbb{S}^B}$). But $\mathbb{S}^{\Delta_B} I; R_B$ is equal to

$$\mathbb{S}^{\Delta_B} \exists_{\pi_{23}} (\mathbb{S}^{\pi_1} I \sqcap \mathbb{S}^{\pi_{12}} \succeq_B \sqcap \mathbb{S}^{\pi_{13}} \succeq_B)$$

(where $\pi_1 : B \times B \times B \to B$) which reduces to

$$\exists_{\pi_2} (\mathbb{S}^{\pi_1} I \sqcap \succeq_B)$$

by Beck–Chevalley applied to the pullback square

$$\begin{array}{ccc} (B\times B)_{\pi_2} & \xrightarrow{\mathrm{Id}_B \times \Delta_B} & B \times B \times B \\ \downarrow & & \downarrow {\scriptstyle \pi_{23}} \\ B & \xrightarrow{\Delta_B} & B \times B \end{array}$$

together with noting that

$$\mathbb{S}^{\mathrm{Id}_B\times\Delta_B}\mathbb{S}^{\pi_1}=\mathbb{S}^{\pi_1}\quad\text{and}\quad\mathbb{S}^{\mathrm{Id}_B\times\Delta_B}\mathbb{S}^{\pi_{12}}=\mathrm{Id}_{\mathbb{S}^{B\times B}}=\mathbb{S}^{\mathrm{Id}_B\times\Delta_B}\mathbb{S}^{\pi_{13}}.$$

But $\exists_{\pi_2}(\mathbb{S}^{\pi_1}I\sqcap\succeq_B)=I;\succeq_B$ by the definition of ;, so we are done checking that $I;R_B\sqsubseteq I\otimes I$ implies that I is lower closed.

Secondly, in the other direction, say I is lower closed, so we have $I;\succeq_B\sqsubseteq I$, i.e. $\exists_{\pi_2}(\mathbb{S}^{\pi_1}I\sqcap\succeq_B)\sqsubseteq I$. Then, since $\exists_{\pi_2}$ is left adjoint to $\mathbb{S}^{\pi_2}$ we have that $\mathbb{S}^{\pi_1}I\sqcap\succeq_B\sqsubseteq\mathbb{S}^{\pi_2}I$. To show that $I;R_B\sqsubseteq I\otimes I$ we are required to show

$$\exists_{\pi_{23}}(\mathbb{S}^{\pi_1}I\sqcap\mathbb{S}^{\pi_{12}}\succeq_B\sqcap\mathbb{S}^{\pi_{13}}\succeq_B)\sqsubseteq\mathbb{S}^{\pi_1}I\sqcap\mathbb{S}^{\pi_2}I.$$

We will just show that

$$\exists_{\pi_{23}}(\mathbb{S}^{\pi_1}I\sqcap\mathbb{S}^{\pi_{12}}\succeq_B\sqcap\mathbb{S}^{\pi_{13}}\succeq_B)\sqsubseteq\mathbb{S}^{\pi_1}I$$

since the proof of the same inequality but with π_2 on the RHS rather than π_1 is symmetric. However this is immediate by noting that $\exists_{\pi_{23}}\dashv\mathbb{S}^{\pi_{23}}$, that $\mathbb{S}^{\pi_{23}}\mathbb{S}^{\pi_1}$ factors as $\mathbb{S}^{\pi_{12}}\mathbb{S}^{\pi_2}$ and that $\mathbb{S}^{\pi_1}I$ (for $\pi_1:B\times B\times B\to B$) is equal to $\mathbb{S}^{\pi_{12}}\mathbb{S}^{\pi_1}I$, since then

$$\begin{aligned}\mathbb{S}^{\pi_1}I\sqcap\mathbb{S}^{\pi_{12}}\succeq_B\sqcap\mathbb{S}^{\pi_{13}}\succeq_B&\sqsubseteq\mathbb{S}^{\pi_{12}}(\mathbb{S}^{\pi_1}I\sqcap\succeq_B)\\&\sqsubseteq\mathbb{S}^{\pi_{12}}\mathbb{S}^{\pi_2}I=\mathbb{S}^{\pi_{23}}\mathbb{S}^{\pi_1}I.\end{aligned}\qquad\square$$

We now relate the construction of the ideal completion of a preorder to a more general construction given in [9]. The reason for checking that our construction is the same is that we will need some of the known properties of the more general construction in our applications.

Our Axiom 8 asserts that P_L is a KZ-monad and [9] shows that this implies that certain inflationary idempotents split. In detail, with an assumption that the lower power monad is KZ, given any object X and an inflationary and idempotent join semilattice homomorphism $\alpha:\mathbb{S}^X\to\mathbb{S}^X$, we can find an object E and join semilattice homomorphisms $\beta:\mathbb{S}^X\to\mathbb{S}^E$ and $\gamma:\mathbb{S}^E\to\mathbb{S}^X$ such that $\gamma\beta=\alpha$ and $\beta\gamma=\mathrm{Id}_{\mathbb{S}^E}$. In fact [9] shows that γ is a distributive lattice homomorphism; this fact will play a key role later. We now recall how E is constructed. Consider the morphisms

$$\varepsilon^{\sqcap}:\mathbb{S}^X\times\mathbb{S}^X\xrightarrow{\Diamond_X\times\Diamond_X}\mathbb{S}^{P_LX}\times\mathbb{S}^{P_LX}\xrightarrow{\sqcap}\mathbb{S}^{P_LX}$$

and

$$\delta^{\sqcap}:\mathbb{S}^X\times\mathbb{S}^X\xrightarrow{\alpha\times\alpha}\mathbb{S}^X\times\mathbb{S}^X\xrightarrow{\sqcap}\mathbb{S}^X\xrightarrow{\Diamond_X}\mathbb{S}^{P_LX}.$$

Let $E^{\sqcap} \hookrightarrow P_L X$ be the equalizer in $\mathcal{C}$ of their double exponential transposes, i.e. of

$$P_L X \underset{\overline{\delta^{\sqcap}}}{\overset{\overline{\epsilon^{\sqcap}}}{\rightrightarrows}} \mathbb{P}(X + X).$$

Further consider the morphisms

$$\varepsilon^1 : 1 \xrightarrow{1} \mathbb{S}^{P_L X} \quad \text{and} \quad \delta^1 : 1 \xrightarrow{1} \mathbb{S}^X \xrightarrow{\Diamond_X} \mathbb{S}^{P_L X},$$

and let $E^1 \hookrightarrow P_L X$ be the equalizer in $\mathcal{C}$ of their exponential transposes. Take $e_X : E \hookrightarrow P_L X$ to be the intersection of E^1 and $E^{\sqcap}$. To check that this construction is indeed a generalisation of the definition of ideal completion of a preorder just given, we need to prove:

Lemma 4.4.5. $E \cong \mathrm{Idl}(B)$ *in the case that* $X = B$*, a preorder relative to* $\mathcal{C}$*, and* $\alpha = \alpha_{\preceq_B}$.

Proof. There is a monomorphism

$$\overline{\Diamond_{B\times B}\otimes} : P_L(B \times B) \hookrightarrow \mathbb{P}(X + X)$$

given by mapping any join semilattice homomorphism $\gamma : \mathbb{S}^{B\times B} \to \mathbb{S}^Y$ to $\mathbb{S}^B \times \mathbb{S}^B \overset{\otimes}{\to} \mathbb{S}^{B\times B} \overset{\gamma}{\to} \mathbb{S}^Y$. The result then follows by verifying that the equalizer diagram given in the construction of E factors as the equalizer diagram given in the definition of $\mathrm{Idl}(B)$ followed by $\overline{\Diamond_{B\times B}\otimes}$. Since $\overline{\Diamond_{B\times B}\otimes}$ is a monomorphism this shows that the two constructions are isomorphic. In detail to complete the proof we need to check that

(A) $\mathbb{S}^B \xrightarrow{\cong} P_L B \xrightarrow{\overline{\varepsilon^{\sqcap}}} \mathbb{P}(B + B)$ factors as

$$\mathbb{S}^B \xrightarrow{\otimes\Delta_{\mathbb{S}^B}} \mathbb{S}^{B\times B} \cong P_L(B \times B) \hookrightarrow \mathbb{P}(X + X),$$

(B) $\mathbb{S}^B \xrightarrow{\cong} P_L B \xrightarrow{\overline{\delta^{\sqcap}}} \mathbb{P}(B + B)$ factors as

$$\mathbb{S}^B \xrightarrow{\alpha_{R_B}} \mathbb{S}^{B\times B} \cong P_L(B \times B) \hookrightarrow \mathbb{P}(X + X),$$

and similarly for the two nullary cases.

To show (A) we check that for any $Y \overset{z}{\to} \mathbb{S}^B$,

$$Y \xrightarrow{z} \mathbb{S}^B \xrightarrow{\cong} P_L B \xrightarrow{\overline{\varepsilon^{\sqcap}}} \mathbb{P}(B + B) \tag{I}$$

is equal to

$$Y \xrightarrow{z} \mathbb{S}^B \xrightarrow{\otimes\Delta_{\mathbb{S}^B}} \mathbb{S}^{B\times B} \cong P_L(B\times B) \hookrightarrow^{\overline{\Diamond_{B\times B}\otimes}} \mathbb{P}(X+X). \tag{II}$$

Since all of the constructions involved are stable under change of base we can reduce to the case that $Y = 1$ and that therefore $z = I$, some open of B. In this case (II) corresponds to the natural transformation

$$\mathbb{S}^B \times \mathbb{S}^B \xrightarrow{\otimes} \mathbb{S}^{B\times B} \xrightarrow{\exists_{B\times B}(I\otimes I\sqcap(\_))} \mathbb{S} \tag{ii}$$

since the map $\overline{\Diamond_{B\times B}\otimes}$ is effectively ‘precompose with $\otimes$’ and the isomorphism $\mathbb{S}^{B\times B} \cong P_L(B\times B)$ sends any relation R on $B\times B$ to $\exists_B(R\sqcap(\_))$. (I) on the other hand corresponds (in the case $Y = 1$, $z = I$) to

$$\mathbb{S}^B \times \mathbb{S}^B \xrightarrow{\Diamond_X\times\Diamond_X} \mathbb{S}^{P_L B}\times\mathbb{S}^{P_L B} \xrightarrow{\sqcap} \mathbb{S}^{P_L B} \xrightarrow{\mathbb{S}^{p_I}} \mathbb{S}$$

which is equal to

$$\mathbb{S}^B \times \mathbb{S}^B \xrightarrow{\exists_B(I\sqcap(\_))\times\exists_B(I\sqcap(\_))} \mathbb{S}\times\mathbb{S} \xrightarrow{\sqcap} \mathbb{S} \tag{i}$$

since $\mathbb{S}^{p_I}$ is a meet semilattice homomorphism and $\mathbb{S}^{p_I}\Diamond_B =$ ‘the join semilattice homomorphism $\mathbb{S}^B \to \mathbb{S}$ corresponding to I’, which is equal to $\exists_B(I\sqcap(\_))$ a result we have already just drawn on. To prove that (i) is equal to (ii) it is sufficient, again by change of base, to check that they are the same when precomposed with $1 \xrightarrow{(J,K)} \mathbb{S}^B\times\mathbb{S}^B$ for arbitrary opens J and K of B. So a verification of (A) reduces to checking that

$$\exists_{B\times B}(I\otimes I\sqcap J\otimes K) = \exists_B(I\sqcap J)\sqcap\exists_B(I\sqcap K). \tag{$*$}$$

But the LHS of $(*)$ is equal to

$$\begin{aligned}
&\exists_B\exists_{\pi_1}(\mathbb{S}^{\pi_1}(I\sqcap J)\sqcap\mathbb{S}^{\pi_2}(I\sqcap K))\\
&\quad = \exists_B[(I\sqcap J)\sqcap\exists_{\pi_1}\mathbb{S}^{\pi_2}(I\sqcap K)]\\
&\quad = \exists_B[(I\sqcap J)\sqcap\mathbb{S}^{!^B}\exists_B(I\sqcap K)]\\
&\quad = \exists_B(I\sqcap J)\sqcap\exists_B(I\sqcap K),
\end{aligned}$$

where the first step is because $\pi_1 : B\times B \to B$ is open (it is the pullback of the open map $!^B : B \to 1$), the second step is Beck–Chevalley applied to the pullback square

$$\begin{array}{ccc} B \times B & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle !^B} \\ B & \xrightarrow{!^B} & 1 \end{array}$$

and the third and final step is because $!^B$ is open. This completes our verification of (A).

To check (B) it is sufficient to show for any $Y \stackrel{z}{\to} \mathbb{S}^B$ that $\overline{\Diamond_{B\times B}\otimes}\alpha_{R_B} z = \overline{\delta^{\sqcap}} z$. By change of base we can take $Y = 1$ and $z = I$, an arbitrary open of B. Now $\overline{\Diamond_{B\times B}\otimes}\alpha_{R_B} I$ is the natural transformation

$$\mathbb{S}^B \times \mathbb{S}^B \xrightarrow{\otimes} \mathbb{S}^{B\times B} \xrightarrow{\exists_{B\times B}(I;R\sqcap(\_))} \mathbb{S}.$$

The image of I under $\mathbb{S}^B \stackrel{\cong}{\to} P_L B \stackrel{\overline{\delta^{\sqcap}}}{\to} \mathbb{P}(B+B)$ is

$$\mathbb{S}^B \times \mathbb{S}^B \xrightarrow{\alpha_{\preceq_B} \times \alpha_{\preceq_B}} \mathbb{S}^B \times \mathbb{S}^B \xrightarrow{\sqcap} \mathbb{S}^B \xrightarrow{\Diamond_B} \mathbb{S}^{P_L(B)} \xrightarrow{\mathbb{S}^{p_I}} \mathbb{S}$$

which is equal to

$$\mathbb{S}^B \times \mathbb{S}^B \xrightarrow{\alpha_{\preceq_B} \times \alpha_{\preceq_B}} \mathbb{S}^B \times \mathbb{S}^B \xrightarrow{\sqcap} \mathbb{S}^B \xrightarrow{\exists_B(I\sqcap(\_))} \mathbb{S}.$$

So, by change of base, to establish (B) it is sufficient to check for any opens $J, K : 1 \to \mathbb{S}^B$ of B that

$$\exists_B(I \sqcap (J;\preceq_B) \sqcap (K;\preceq_B)) = \exists_{B\times B}(I; R_B \sqcap \mathbb{S}^{\pi_1} J \sqcap \mathbb{S}^{\pi_2} K). \qquad (*)$$

Since $R_B = \mathbb{S}^{\pi_{12}} \succeq_B \sqcap \mathbb{S}^{\pi_{13}} \succeq_B$ this becomes an application of Beck–Chevalley together with various applications of the definition of a map being open, since all the projections involved are open as they are pullbacks of the open map $!^B : B \to 1$. The RHS of $(*)$ is equal to

$$\begin{aligned} &\exists_{B\times B}(\exists_{\pi_{23}}(\mathbb{S}^{\pi_1} I \sqcap R_B) \sqcap \mathbb{S}^{\pi_1} J \sqcap \mathbb{S}^{\pi_2} K) \\ &\quad = \exists_{B\times B}(\exists_{\pi_{23}}(\mathbb{S}^{\pi_1} I \sqcap \mathbb{S}^{\pi_{12}} \succeq_B \sqcap \mathbb{S}^{\pi_{13}} \succeq_B) \sqcap \mathbb{S}^{\pi_1} J \sqcap \mathbb{S}^{\pi_2} K) \\ &\quad = \exists_{B\times B}\exists_{\pi_{23}}((\mathbb{S}^{\pi_1} I \sqcap \mathbb{S}^{\pi_{12}} \succeq_B \sqcap \mathbb{S}^{\pi_{13}} \succeq_B) \sqcap \mathbb{S}^{\pi_2} J \sqcap \mathbb{S}^{\pi_3} K) \end{aligned}$$

since $\pi_2 : B \times B \times B \to B$ factors as $B \times B \times B \stackrel{\pi_{23}}{\to} B \times B \stackrel{\pi_1}{\to} B$ and $\pi_3 : B \times B \times B \to B$ factors as $B \times B \times B \stackrel{\pi_{23}}{\to} B \times B \stackrel{\pi_2}{\to} B$.

The LHS of $(*)$ is equal to

$$\begin{aligned}
&\exists_B(I \sqcap \exists_{\pi_1}(\mathbb{S}^{\pi_2}J \sqcap \succeq_B) \sqcap \exists_{\pi_1}(\mathbb{S}^{\pi_2}K \sqcap \succeq_B)) \\
&\quad = \exists_B\exists_{\pi_1}(\mathbb{S}^{\pi_1}I \sqcap (\mathbb{S}^{\pi_2}J \sqcap \succeq_B) \sqcap \mathbb{S}^{\pi_1}\exists_{\pi_1}(\mathbb{S}^{\pi_2}K \sqcap \succeq_B)) \\
&\quad = \exists_B\exists_{\pi_1}(\mathbb{S}^{\pi_1}I \sqcap (\mathbb{S}^{\pi_2}J \sqcap \succeq_B) \sqcap \exists_{\pi_{12}}\mathbb{S}^{\pi_{13}}(\mathbb{S}^{\pi_2}K \sqcap \succeq_B)) \\
&\quad = \exists_B\exists_{\pi_1}\exists_{\pi_{12}}(\mathbb{S}^{\pi_1}I \sqcap \mathbb{S}^{\pi_2}J \sqcap \mathbb{S}^{\pi_{12}} \succeq_B \sqcap \mathbb{S}^{\pi_3}K \sqcap \mathbb{S}^{\pi_{12}} \succeq_B),
\end{aligned}$$

where the second last line is by Beck–Chevalley on the pullback square

$$\begin{array}{ccc}
B \times B \times B & \xrightarrow{\pi_{13}} & B \times B \\
{\scriptstyle \pi_{12}}\downarrow & & \downarrow{\scriptstyle \pi_1} \\
B \times B & \xrightarrow{\pi_1} & B
\end{array}$$

This completes the proof of (B) since $\exists_B\exists_{\pi_1}\exists_{\pi_{12}} = \exists_{B\times B}\exists_{\pi_{23}}$ as both are left adjoint to $\mathbb{S}^{!^{B\times B\times B}} : \mathbb{S} \to \mathbb{S}^{B\times B\times B}$.

Next we check the nullary cases.

To check that $\exists_B : \mathbb{S}^B \to \mathbb{S}$ factors as $\mathbb{S}^B \cong P_L(B) \xrightarrow{\overline{\delta^1}} \mathbb{S}$, by change of base we are reduced to checking that

$$\exists_B(I) = \overline{\delta^1}(J \mapsto \exists_B(I \sqcap J))$$

for any open I of B. But $\overline{\delta^1} : P_L(B) \xrightarrow{\overline{\delta^1}} \mathbb{S}$ is the map which sends any $\alpha : \mathbb{S}^B \to \mathbb{S}$ to $\alpha(1_{\mathbb{S}^B})$ since it is the exponential transpose of

$$1 \xrightarrow{1_{\mathbb{S}^B}} \mathbb{S}^B \xrightarrow{\diamond_B} \mathbb{S}^{P_L(B)}$$

and so is equal to

$$P_L(B) \hookrightarrow \mathbb{S}^{\mathbb{S}^B} \xrightarrow{\mathbb{S}^{1_{\mathbb{S}^B}}} \mathbb{S}.$$

Therefore $\overline{\delta^1}(J \mapsto \exists_B(I \sqcap J)) = \exists_B(I \sqcap 1_{\mathbb{S}^B}) = \exists_B(I)$ and we are done checking that $\exists_B$ factors via $\overline{\delta^1}$.

That $\mathbb{S}^B \xrightarrow{1_{\mathbb{S}}!^{\mathbb{S}^B}} \mathbb{S}$ factors as $\mathbb{S}^B \cong P_L(B) \xrightarrow{\overline{\varepsilon^1}} \mathbb{S}$ is immediate since $\overline{\varepsilon^1}$ factors as $P_L(B) \xrightarrow{!^{P_L(B)}} 1 \xrightarrow{1_{\mathbb{S}}} \mathbb{S}$ by definition of $1_{\mathbb{S}^{P_L(B)}}$, so we are done checking the nullary cases and have finished the proof. □

In other words, we have now checked that the more general construction of the ideal completion as given, in [9], for any X with an inflationary idempotent join semilattice $\alpha : \mathbb{S}^X \to \mathbb{S}^X$ coincides with our $\mathrm{Idl}(B)$ for the case that X is a preorder B.

4.5 The ideal completion of a join semilattice

In this subsection we make the entirely routine observation that if B is an internal join semilattice in $\mathsf{Dis}_{\mathcal{C}}$, and is therefore a preorder relative to $\mathcal{C}$, then the construction of $\mathrm{Idl}(B)$ can be simplified to an equalizer involving the join operation and no longer needs to be explicitly dependent on relational composition. This allows us to easily make the construction Idl(_) functorial and to give an explicit description of arbitrary maps $Y \to \mathrm{Idl}(B)$ in $\mathcal{C}$. It is the functoriality of this construction and the explicit description of its general points that is the key to our proof, to follow in the final section, of Joyal and Tierney's result about the logical slice stability of locales.

We use the notation $\vee\text{-}\mathsf{SLat}_{\mathsf{Dis}_{\mathcal{C}}}$ to denote the category of internal join semilattices in $\mathsf{Dis}_{\mathcal{C}}$. The '$\vee$' is just a notational convenience to help us decide the direction of the ordering on any given semilattice. For any internal join semilattice in a cartesian category, say denoted by $(B, \vee_B, 0_B)$, as usual we define $\geq_B \hookrightarrow B \times B$ to be the equalizer of

$$B \times B \overset{\vee_B}{\underset{\pi_1}{\rightrightarrows}} B,$$

and as in our discussion of preorders we define $R_B \hookrightarrow B \times B \times B$ to be the intersection $\pi_{12}^{*}{\geq_B}$ and $\pi_{13}^{*}{\geq_B}$. In fact R_B has an alternative description in this case:

Lemma 4.5.1. *For any join semilattice $(B, \vee_B, 0_B)$ in a cartesian category, R_B is isomorphic to the pullback of $\geq_B \hookrightarrow B \times B$ along $\mathrm{Id}_B \times \vee_B : B \times B \times B \to B \times B$.*

Proof. Intuitively, this is just the assertion that the set $\{(b_1, b_2, b_3) \mid b_1 \geq_B b_2$ and $b_1 \geq_B b_3\}$ is equal to $\{(b_1, b_2, b_3) \mid b_1 \geq_B b_2 \vee_B b_3\}$ which of course is entirely trivial. The categorical verification that the two pullbacks are isomorphic is routine from the definition of being an internal join semilattice and the definition of $\geq_B$ as an equalizer. Although the proof is entirely routine the diagram chase involved is slightly lengthy. □

So, for join semilattices in $\mathsf{Dis}_{\mathcal{C}}$, we have $R_B = \mathbb{S}^{\mathrm{Id}_B \times \vee_B}(\geq_B)$.

Lemma 4.5.2. *For any join semilattice $(B, \vee_B, 0_B)$ in $\mathsf{Dis}_{\mathcal{C}}$, $\mathrm{Idl}(B)$ is given by the intersection of the equalizer of*

$$\mathbb{S}^B \overset{\mathbb{S}^{\vee_B}}{\underset{\otimes \Delta_{\mathbb{S}^B}}{\rightrightarrows}} \mathbb{S}^{B \times B}$$

and the equalizer of

$$\mathbb{S}^B \underset{1_{\mathbb{S}}!^{\mathbb{S}^B}}{\overset{\mathbb{S}^{0_B}}{\rightrightarrows}} \mathbb{S}.$$

Proof. We verify for arbitrary $Y \xrightarrow{z} \mathbb{S}^B$ that the morphisms

$$Y \xrightarrow{z} \mathbb{S}^B \underset{(\otimes\Delta_{\mathbb{S}^B}, 1_{\mathbb{S}}!^{\mathbb{S}^B})}{\overset{(\mathbb{S}^{\vee_B}, \mathbb{S}^{0_B})}{\rightrightarrows}} \mathbb{S}^{B\times B} \times \mathbb{S} \qquad (\alpha)$$

are equal if and only if the morphisms

$$Y \xrightarrow{z} \mathbb{S}^B \underset{(\otimes\Delta_{\mathbb{S}^B}, 1_{\mathbb{S}}!^{\mathbb{S}^B})}{\overset{(\alpha_{R_B}, \exists_B)}{\rightrightarrows}} \mathbb{S}^{B\times B} \times \mathbb{S} \qquad (\beta)$$

are equal. As usual, by change of base we can assume that $Y = 1$ and $z = I$, an open of B.

$(\beta) \Rightarrow (\alpha)$. If (β) is true then I is lower closed by Proposition 4.4.4. We firstly check that $\alpha_{R_B}(I) = \mathbb{S}^{\vee_B}(I)$. We have just recalled that $R_B = \mathbb{S}^{\mathrm{Id}_B \times \vee_B}(\geq_B)$ and so,

$$\begin{aligned}
\alpha_{R_B}(I) &= \exists_{\pi_{23}}(R_B \sqcap \mathbb{S}^{\pi_1}(I)) \\
&= \exists_{\pi_{23}}(\mathbb{S}^{\mathrm{Id}_B \times \vee_B}(\geq_B) \sqcap \mathbb{S}^{\pi_1}(I)) \\
&= \exists_{\pi_{23}}(\mathbb{S}^{\mathrm{Id}_B \times \vee_B}(\geq_B) \sqcap \mathbb{S}^{\mathrm{Id}_B \times \vee_B}\mathbb{S}^{\pi_1}(I)) \\
&= \exists_{\pi_{23}}(\mathbb{S}^{\mathrm{Id}_B \times \vee_B}[(\geq_B) \sqcap \mathbb{S}^{\pi_1}(I)]) \\
&= \mathbb{S}^{\vee_B}\exists_{\pi_2}[(\geq_B) \sqcap \mathbb{S}^{\pi_1}(I)] \\
&= \mathbb{S}^{\vee_B}(I),
\end{aligned}$$

where the second last line is by Beck–Chevalley on the pullback square

$$\begin{array}{ccc}
B \times B \times B & \xrightarrow{\mathrm{Id}_B \times \vee_B} & B \times B \\
{\scriptstyle \pi_{23}}\downarrow & & \downarrow{\scriptstyle \pi_2} \\
B \times B & \xrightarrow{\vee_B} & B
\end{array}$$

and the last line is because I is lower closed.

Next we check that $\mathbb{S}^{0_B}(I) = \exists_B(I)$;

$$\begin{aligned}\mathbb{S}^{0_B}(I) &= \mathbb{S}^{0_B}(\exists_{\pi_2}(\geq_B \sqcap \mathbb{S}^{\pi_1} I)) \\ &= \exists_B \mathbb{S}^{(\mathrm{Id}_B, 0_B !^B)}(\geq_B \sqcap \mathbb{S}^{\pi_1} I) \\ &= \exists_B(I),\end{aligned}$$

where the second line is by Beck–Chevalley applied to the pullback square

$$\begin{array}{ccc} B & \xrightarrow{(\mathrm{Id}_B, 0_B)} & B \times B \\ {\scriptstyle !^B}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ 1 & \xrightarrow{0_B} & B \end{array}$$

and the last line is because $\mathbb{S}^{(\mathrm{Id}_B, 0_B !^B)}(\geq_B) = 1_{\mathbb{S}^B}$ since the pullback of $\geq_B \hookrightarrow B \times B$ along $B \stackrel{(\mathrm{Id}_B, 0_B)}{\longrightarrow} B \times B$ is isomorphic to the identity on B. This completes our proof of $(\beta) \Rightarrow (\alpha)$.

$(\alpha) \Rightarrow (\beta)$. Assume (α). Then $\mathbb{S}^{0_B}(I) = 1_{\mathbb{S}}$. By Lemma 4.1.4 to prove that $\exists_B(I) = 1_{\mathbb{S}}$ it is sufficient to check that $\ulcorner \Delta_B \urcorner 0_B \sqsubseteq I$. The open $\ulcorner \Delta_B \urcorner 0_B$ of B is the regular monomorphism $1 \stackrel{0_B}{\rightarrow} B$ since we have a pullback square

$$\begin{array}{ccc} 1 & \xrightarrow{0_B} & B \\ {\scriptstyle 0_B}\downarrow & & \downarrow{\scriptstyle \Delta_B} \\ B & \xrightarrow{(\mathrm{Id}_B, 0_B !^B)} & B \times B \end{array}$$

But $\mathbb{S}^{0_B}(I) = 1_{\mathbb{S}}$ implies $1_{\mathbb{S}} : 1 \rightarrow \mathbb{S}$ factors as $1 \stackrel{0_B}{\rightarrow} B \stackrel{I}{\rightarrow} \mathbb{S}$, and so for any $W \stackrel{z}{\rightarrow} B$, if it factors via $1 \stackrel{0_B}{\rightarrow} B$ then it also factors via I; this shows that $\ulcorner \Delta_B \urcorner 0_B \sqsubseteq I$ (see the comments after the introduction of Axiom 3) and so $\exists_B(I) = 1_{\mathbb{S}}$.

Finally, we show that if $\mathbb{S}^{\vee_B}(I) = I \otimes I$ then $\alpha_{R_B}(I) = I \otimes I$. However the proof shown in $(\beta) \Rightarrow (\alpha)$, that $\mathbb{S}^{\vee_B}(I) = \alpha_{R_B}(I)$, will be available provided we can check that (α) implies that I is lower closed. To prove that I is lower closed we check $\exists_{\pi_2}(\mathbb{S}^{\pi_1} I \sqcap \geq_B) \sqsubseteq I$, which is equivalent to checking $\mathbb{S}^{\pi_1} I \sqcap \geq_B \sqsubseteq \mathbb{S}^{\pi_2} I$. But $\mathbb{S}^{\vee_B}(I) = I \otimes I = \mathbb{S}^{\pi_1} I \sqcap \mathbb{S}^{\pi_2} I \sqsubseteq \mathbb{S}^{\pi_2} I$ so it is sufficient to check that

$$\mathbb{S}^{\pi_1} I \sqcap \geq_B \sqsubseteq \mathbb{S}^{\vee_B} I.$$

Now for any $W \stackrel{z}{\rightarrow} B$ if z factors through both (i) $\mathbb{S}^{\pi_1} I$ and (ii) $\geq_B$ then (i) $I \pi_1 z = 1_{\mathbb{S}} !^{B \times B} z$ and (ii) $\vee_B z = \pi_1 z$. Putting these together we get that

$$I \vee_B z = 1_{\mathbb{S}} !^{B \times B} z$$

and so z factors through $\mathbb{S}^{\vee_B} I$. This shows that $\mathbb{S}^{\pi_1} I \sqcap \geq_B \sqsubseteq \mathbb{S}^{\vee_B} I$ and we are done. □

Lemma 4.5.3. *The construction given of* $\mathrm{Idl}(B)$ *extends to a functor* $\mathrm{Idl} : \vee\text{-}\mathsf{SLat}_{\mathsf{Dis}_{\mathcal{C}}} \to \mathcal{C}^{\mathrm{op}}$ *which is a subfunctor of* $\mathbb{S}^{(\_)}$.

Proof. Say $f : B_1 \to B_2$ is a join semilattice homomorphism. Then it is clear that we have the following commuting diagram:

$$\begin{array}{ccccc} \mathrm{Idl}(B_2) & \hookrightarrow & \mathbb{S}^{B_2} & \rightrightarrows & \mathbb{S}^{B_2 \times B_2} \times \mathbb{S} \\ & & \downarrow{\scriptstyle \mathbb{S}^f} & & \downarrow{\scriptstyle \mathbb{S}^{f \times f} \times \mathrm{Id}_{\mathbb{S}}} \\ \mathrm{Idl}(B_1) & \hookrightarrow & \mathbb{S}^{B_1} & \rightrightarrows & \mathbb{S}^{B_1 \times B_1} \times \mathbb{S} \end{array}$$

and so there exists unique $\mathrm{Idl}(f) : \mathrm{Idl}(B_2) \to \mathrm{Idl}(B_1)$ making a commuting square on the left. □

The following lemma clarifies why we have introduced the ideal completion of semilattices as it provides a description of the general points of $\mathrm{Idl}(B)$.

Lemma 4.5.4. *There are natural isomorphisms:*

(i) $\mathcal{C}(W, \mathrm{Idl}(B)) \cong \{f : B \to \mathbb{S}^W \mid f$ *takes finite joins to meets*$\}$ *for any join semilattice* B *and any object* W*, and*

(ii) $\mathcal{C}(W, \mathrm{Idl}(B^{\mathrm{op}})) \cong \wedge\text{-}\mathsf{SLat}(B \to \mathbb{S}^W)$ *for any meet semilattice* B *and any object* W.

Proof. (i) is immediate from the construction. (ii) is the same assertion as (i) since B is a meet semilattice if and only if B^{op} is a join semilattice. □

Note that (ii) is, essentially, the well-known observation from lattice theory that $\mathcal{D}B$, the set of lower closed subsets of B, is the free frame on the meet semilattice B. The reason for introducing (ii) is that in application we will want to view frames as (infinitary) algebras over meet semilattices in the familiar way. Of course we could have introduced the filter completion of a preorder, dual to the ideal completion, and so avoided the need to introduce opposites at this point. However, although the filter completion has exactly the same mathematical content as the ideal completion, it is less familiar and so we have chosen not to introduce it.

We end this subsection with a couple of technical lemmas. They are applications of the description just given of the points of $\mathrm{Idl}(B^{\mathrm{op}})$ and are required for the final section.

Lemma 4.5.5. *If A and B are two internal meet semilattices with A and B discrete objects of $\mathcal{C}$ then the following assertions hold:*

(i) $\mathcal{C}(\mathrm{Idl}(A^{\mathrm{op}}), \mathrm{Idl}(B^{\mathrm{op}})) \cong \{B \xrightarrow{f} \mathbb{S}^A \mid f$ *a* $\wedge$-SLat *hom.*, $\alpha_{\geq_A} f = f\}$.

(ii) *If $A = B$ then the image of the identity is* $\ulcorner \geq_B \urcorner : B \to \mathbb{S}^B$.

(iii) *If $h : B \to A$ is a meet semilattice homomorphism then the image of* $\mathrm{Idl}(h)^{\mathrm{op}} : \mathrm{Idl}(A^{\mathrm{op}}) \to \mathrm{Idl}(B^{\mathrm{op}})$ *under* (i) *is*

$$B \xrightarrow{\ulcorner (h \times \mathrm{Id}_A)^*(\geq_A) \urcorner} \mathbb{S}^A.$$

Proof. The order isomorphism follows by examining the case $W = \mathrm{Idl}(A^{\mathrm{op}})$ in $\mathcal{C}(W, \mathrm{Idl}(B^{\mathrm{op}})) \cong \wedge\text{-SLat}(B \to \mathbb{S}^W)$ and recalling that $\mathbb{S}^{\mathrm{Idl}(A^{\mathrm{op}})}$ is the split equalizer of

$$\mathbb{S}^A \underset{\alpha_{\geq_A}}{\overset{\mathrm{Id}}{\rightrightarrows}} \mathbb{S}^A.$$

This splitting,

$$\mathbb{S}^{\mathrm{Idl}(A^{\mathrm{op}})} \underset{\delta_A}{\overset{\varepsilon_A}{\rightleftarrows}} \mathbb{S}^A$$

say, has the additional properties that

(i) ε_A is a distributive lattice homomorphism, and

(ii) $\delta_A = \mathbb{S}^{e_A} \Diamond_A$, where $e_A : \mathrm{Idl}(A^{\mathrm{op}}) \hookrightarrow P_L A$ as introduced before Lemma 4.4.5.

(See (ii) $\Rightarrow$ (iii) of [9, Theorem 2] for the order dual observations.)

From (i) we get that composition with ε_A defines an order isomorphism between meet semilattice homomorphisms $B \to \mathbb{S}^{\mathrm{Idl}(A^{\mathrm{op}})}$ and meet semilattice homomorphisms $f : B \to \mathbb{S}^A$ such that $\alpha_{\geq_A} f = f$. This completes our proof of the assertion (i).

To prove that the image of the identity on $\mathrm{Idl}(B^{\mathrm{op}})$ is $\ulcorner \geq_B \urcorner : B \to \mathbb{S}^B$ we must check that

$$B \xrightarrow{\widetilde{i_B}} \mathbb{S}^{\mathrm{Idl}(B^{\mathrm{op}})} \xhookrightarrow{\varepsilon_B} \mathbb{S}^B$$

is equal to $\ulcorner \geq_B \urcorner : B \to \mathbb{S}^B$ where $\widetilde{i_B}$ is the double exponential transpose of $\mathrm{Idl}(B^{\mathrm{op}}) \overset{i_B}{\hookrightarrow} \mathbb{S}^B$. Since $\alpha_{\geq_B}$ factors as $\varepsilon_B \delta_B = \varepsilon_B \mathbb{S}^{e_B} \Diamond_B$ and $\alpha_{\geq_B} \ulcorner \Delta_B \urcorner =$

$\ulcorner \geq_B \urcorner$ it is sufficient, to complete the proof of (ii), to check that $\widetilde{i_B} = \mathbb{S}^{e_B} \Diamond_B \ulcorner \Delta_B \urcorner$. But e_B is related to i_B by

$$\begin{array}{ccc} \mathrm{Idl}(B^{\mathrm{op}}) & \xhookrightarrow{e_B} & P_L(B) \\ & \searrow^{i_B} & \downarrow \cong \\ & & \mathbb{S}^B \end{array}$$

where the isomorphism $P_L(B) \to \mathbb{S}^B$ is given by the double exponential transpose of $B \xrightarrow{\ulcorner \Delta_B \urcorner} \mathbb{S}^B \xrightarrow{\Diamond_B} \mathbb{S}^{P_L(B)}$ (see Lemma 4.2.5) and so we are done.

For (iii) note that by the definition of the functor Idl the image of $\mathrm{Idl}(h)^{\mathrm{op}}$: $\mathrm{Idl}(A^{\mathrm{op}}) \to \mathrm{Idl}(B^{\mathrm{op}})$ is equal to the double exponential transpose of

$$\mathrm{Idl}(A^{\mathrm{op}}) \xhookrightarrow{i_A} \mathbb{S}^A \xrightarrow{\mathbb{S}^h} \mathbb{S}^B$$

followed by $\mathbb{S}^{\mathrm{Idl}(A^{\mathrm{op}})} \xhookrightarrow{\varepsilon_A} \mathbb{S}^A$, i.e. equal to

$$B \xrightarrow{h} A \xrightarrow{\widetilde{i_A}} \mathbb{S}^{\mathrm{Idl}(A^{\mathrm{op}})} \xhookrightarrow{\varepsilon_A} \mathbb{S}^A.$$

But we have already observed that $\varepsilon_A \widetilde{i_A} = \ulcorner \geq_A \urcorner$ and so this effectively completes the proof since the exponential transpose of $\ulcorner \geq_A \urcorner h$ classifies the open $(h \times \mathrm{Id}_A)^*(\geq_A)$. □

Notation 4.5.6. We have established a relationship between maps $\mathrm{Idl}(A^{\mathrm{op}}) \to \mathrm{Idl}(B^{\mathrm{op}})$ of $\mathcal{C}$ and open relations on $B \times A$. We will follow a notation that if R is such a relation then n_R is the corresponding map from $\mathrm{Idl}(A^{\mathrm{op}})$ to $\mathrm{Idl}(B^{\mathrm{op}})$.

Lemma 4.5.7. *With the notation just introduced,*

(a) *for any* $n_R : \mathrm{Idl}(A^{\mathrm{op}}) \to \mathrm{Idl}(B^{\mathrm{op}})$

$$\begin{array}{ccc} \mathbb{S}^{\mathrm{Idl}(B^{\mathrm{op}})} & \xrightarrow{\mathbb{S}^{n_R}} & \mathbb{S}^{\mathrm{Idl}(A^{\mathrm{op}})} \\ \downarrow \varepsilon_B & & \downarrow \varepsilon_A \\ \mathbb{S}^B & \xrightarrow{\alpha_R} & \mathbb{S}^A \end{array}$$

commutes, and

(b) *given* $n_R : \mathrm{Idl}(A^{\mathrm{op}}) \to \mathrm{Idl}(B^{\mathrm{op}})$ *and* $n_{R'} : \mathrm{Idl}(B^{\mathrm{op}}) \to \mathrm{Idl}(C^{\mathrm{op}})$

$$n_{R'} n_R = n_{R';R}.$$

Proof. (a) Firstly,

$$\ulcorner R \urcorner = \varepsilon_A \mathbb{S}^{n_R} \widetilde{i_B} = \varepsilon_A \mathbb{S}^{n_R} \delta_B \ulcorner \Delta_B \urcorner,$$

the first step by construction of n_R and the second since $\delta_B \ulcorner \Delta_B \urcorner = \widetilde{i_B}$ was established in the previous lemma. Therefore $\varepsilon_A \mathbb{S}^{n_R} \delta_B = \alpha_R$ by Lemma 4.2.5. So the result follows since $\delta_B \varepsilon_B = \mathrm{Id}$.

(b) Under the order isomorphism (i) of the previous lemma the map $C \to \mathbb{S}^A$ corresponding to $n_{R'} n_R$ is given by the double exponential transpose of

$$\mathrm{Idl}(A^{\mathrm{op}}) \xrightarrow{n_R} \mathrm{Idl}(B^{\mathrm{op}}) \xrightarrow{n_{R'}} \mathrm{Idl}(C^{\mathrm{op}}) \xhookrightarrow{i_C} \mathbb{S}^C$$

followed by $\mathbb{S}^{\mathrm{Idl}(A^{\mathrm{op}})} \xhookrightarrow{\varepsilon_A} \mathbb{S}^A$. Using part (a) the map $C \to \mathbb{S}^A$ corresponding to $n_{R'} n_R$ is therefore given by

$$C \xrightarrow{\widetilde{i_C}} \mathbb{S}^{\mathrm{Idl}(C^{\mathrm{op}})} \xrightarrow{\mathbb{S}^{n_{R'}}} \mathbb{S}^{\mathrm{Idl}(B^{\mathrm{op}})} \xhookrightarrow{\varepsilon_B} \mathbb{S}^B \xrightarrow{\alpha_R} \mathbb{S}^A$$

which is equal to $C \xrightarrow{\ulcorner R' \urcorner} \mathbb{S}^B \xrightarrow{\alpha_R} \mathbb{S}^A$, i.e. to $\ulcorner R'; R \urcorner$ as required. □

5 Applications

5.1 The fundamental theorem of topos theory

Before stating and proving the fundamental theorem of topos theory we recall some facts about local homeomorphisms.

Definition 5.1.1. A morphism $f : X \to Y$ in $\mathcal{C}$ is a *local homeomorphism* provided it is open and the diagonal $X \to X \times_Y X$ is also open.

When $\mathcal{C} = \mathsf{Loc}$ the usual definition for local homeomorphism is that X has an open cover $(a_i)_{i \in I}$ such that each $a_i \hookrightarrow X \xrightarrow{f} Y$ is isomorphic to an open of Y. However our definition is equivalent (see, e.g. [4, Lemma C3.1.15]) and is also a good translation of the topological situation since a map between topological spaces is a local homeomorphism if and only if it is open and its diagonal is open.

For any object X in $\mathcal{C}$ we have that $\mathsf{LH}/X = \mathsf{Dis}_{\mathcal{C}/X}$ (see [8] for the axiomatic proof). We will also need the following easy extension of this observation:

Lemma 5.1.2. *If A is a discrete object of $\mathcal{C}$ then* $\mathsf{Dis}_{\mathcal{C}/A} = (\mathsf{Dis}_{\mathcal{C}})/A$.

Proof. Since $\mathsf{LH}/A = \mathsf{Dis}_{\mathcal{C}/A}$, we must check two things: (i) for any local homeomorphism $f : Y \to A$, Y is necessarily discrete, and (ii) for any discrete B and for any map $f : B \to A$, f is necessarily a local homeomorphism.

For (i), $!^Y : Y \xrightarrow{f} A \xrightarrow{!^A} 1$ must be open since both f and $!^A$ are, the latter by assumption that A is discrete. The inclusion $i : Y \times_A Y \hookrightarrow Y \times Y$ is open since it is the pullback (along $Y \times Y \xrightarrow{f \times f} A \times A$) of the open diagonal $A \hookrightarrow A \times A$. But then the diagonal $Y \hookrightarrow Y \times Y$ must be open since it is the composition of the open $Y \to Y \times_A Y$ (which is open by assumption that f is a homeomorphism) and i. It follows that Y is discrete.

For (ii) firstly f must be open since it is a morphism between discrete objects. But the diagonal $B \to B \times_A B$ is open since it is the pullback (along $i : B \times_A B \hookrightarrow B \times B$) of the open diapasonal $B \to B \times B$. This proves that f is a local homeomorphism. □

For a topos $\mathcal{E}$ (by which we mean an elementary topos), $\mathcal{E}$ is equivalent to the category of discrete locales over $\mathcal{E}$. In fact the inclusion $\mathcal{E} \hookrightarrow \mathsf{Loc}_{\mathcal{E}}$ has a right adjoint: send any locale X to the object $\mathsf{Loc}_{\mathcal{E}}(1, X)$ (i.e. the object of points of X). What is remarkable (and originally observed by Paul Taylor as part of his Abstract Stone Duality programme) is that, in the other direction, if the inclusion $\mathsf{Dis}_{\mathcal{C}} \hookrightarrow \mathcal{C}$ has a right adjoint then $\mathsf{Dis}_{\mathcal{C}}$ is a topos.

Proposition 5.1.3. *If the inclusion* $\mathsf{Dis}_{\mathcal{C}} \hookrightarrow \mathcal{C}$ *has a right adjoint then* $\mathsf{Dis}_{\mathcal{C}}$ *is a topos.*

We will use the notation $(\_)_d : \mathcal{C} \to \mathsf{Dis}_{\mathcal{C}}$ for such a right adjoint if it exists.

Proof. A category is a topos provided it is cartesian and has power objects. We have commented already that $\mathsf{Dis}_{\mathcal{C}}$ is cartesian and so we are left checking that $\mathsf{Dis}_{\mathcal{C}}$ has power objects. For any discrete object A, let PA denote the object $(\mathbb{S}^A)_d$ of $\mathsf{Dis}_{\mathcal{C}}$. To prove that this defines a power object of A we must find a monomorphism $\in_A \hookrightarrow PA \times A$ such that for any monomorphism $R \hookrightarrow B \times A$ there exists a unique map $r : B \to PA$ such that R is the pullback of $\in_A$ along $r \times \mathrm{Id}_A$. Now all morphisms in $\mathsf{Dis}_{\mathcal{C}}$ are open and all monomorphisms in $\mathsf{Dis}_{\mathcal{C}}$ are regular. The open regular monomorphisms in $\mathcal{C}$ are exactly the open subobjects, and so $\mathrm{Sub}_{\mathsf{Dis}_{\mathcal{C}}}(B \times A) \cong \mathcal{C}(B \times A, \mathbb{S})$. So we have that

$$\mathsf{Dis}_{\mathcal{C}}(B, PA) \cong \mathcal{C}(B, \mathbb{S}^A) \cong \mathcal{C}(B \times A, \mathbb{S}) \cong \mathrm{Sub}_{\mathsf{Dis}_{\mathcal{C}}}(B \times A).$$

This bijection is natural in B, so if $\in_A$ is defined as the mate of Id_{PA} then any monomorphism $R \hookrightarrow B \times A$ is the pullback of $\in_A$ along $r \times \mathrm{Id}_A$, where $r : B \to PA$ is the mate of R under the bijection. □

The fundamental theorem of topos theory now becomes a categorical triviality:

Theorem 5.1.4. *If $\mathcal{E}$ is an elementary topos and A is an object of $\mathcal{E}$ then $\mathcal{E}/A$ is an elementary topos.*

Proof. Firstly, as remarked before the last proposition, as $\mathcal{E}$ is a topos, $\mathcal{E} \hookrightarrow \mathsf{Loc}_{\mathcal{E}}$ has a right adjoint. Now $\mathsf{Dis}_{\mathsf{Loc}_{\mathcal{E}}/A} \simeq \mathcal{E}/A$ by the lemma and so it is sufficient to find a right adjoint to the inclusion $\mathcal{E}/A \hookrightarrow \mathsf{Loc}_{\mathcal{E}}/A$. Define $(\_)_{d_A} : \mathsf{Loc}_{\mathcal{E}}/A \to \mathcal{E}/A$ by sending an object $f : X \to A$ of $\mathsf{Loc}_{\mathcal{E}}/A$ to $(f)_d : X_d \to A_d \cong A$. It is easy to verify that $(\_)_{d_A}$ is the required right adjoint. □

For the next section we will need the following basic lemma which is showing that if $(\_)_d$ exists right adjoint to the inclusion $i : \mathsf{Dis}_{\mathcal{C}} \hookrightarrow \mathcal{C}$ then the induced adjoint transpose can commute with exponential transpose (for functions to power sets at least). Recall, of course, that a power set PA can be written as the exponential Ω^A where Ω is the subobject classifier (i.e. $P1$). For toposes we use the standard notation $\chi_I : A \to \Omega$ for the map that classifies a monomorphism $I \hookrightarrow A$. This is in contrast to our notation for $\mathcal{C}$ where we have not always distinguished between $a \hookrightarrow X$, an open regular monomorphism, and $a : X \to \mathbb{S}$.

Lemma 5.1.5. *If $i : \mathsf{Dis}_{\mathcal{C}} \hookrightarrow \mathcal{C}$ has a right adjoint then the following assertions hold:*

(i) *For any objects Y of $\mathcal{C}$ and A of $\mathsf{Dis}_{\mathcal{C}}$, $Y_d^A \cong (Y^A)_d$.*

(ii) *If $R \hookrightarrow C \times B$ is a monomorphism in $\mathsf{Dis}_{\mathcal{C}}$ then the mate of $C \xrightarrow{\ulcorner \chi_R \urcorner} \mathbb{S}_d^B \cong (\mathbb{S}^B)_d$ under the adjunction $i \dashv (\_)_d$ is $C \xrightarrow{\ulcorner R \urcorner} \mathbb{S}^B$ where $R : C \times B \to \mathbb{S}$ classifies the open R.*

(iii) *The counit at $\mathbb{S}$ of the adjunction $i \dashv (\_)_d$ is the map $\top : (\mathbb{S})_d \to \mathbb{S}$ that classifies the open 'true', i.e. the top element of the subobject classifier $(\mathbb{S})_d$.*

Proof. (i) This is a routine diagram chase given that finite products of $\mathsf{Dis}_{\mathcal{C}}$ are created in $\mathcal{C}$.

(ii) The mate of $C \xrightarrow{\ulcorner R \urcorner} \mathbb{S}^B$ under the adjunction is $C \xrightarrow{\ulcorner R \urcorner_d} (\mathbb{S}^B)_d$ (passing through the isomorphism $C \cong C_d$ without notation). Under the isomorphism of part (i) the exponential transpose of

$$C \xrightarrow{\ulcorner R \urcorner_d} (\mathbb{S}^B)_d \cong \mathbb{S}_d^B$$

is $C \times B \xrightarrow{R_d} \mathbb{S}_d$. But $(\_)_d$ preserves the pullback

$$\begin{array}{ccc} R & \longrightarrow & 1 \\ \downarrow & & \downarrow 1_{\mathbb{S}} \\ C \times B & \xrightarrow{R} & \mathbb{S} \end{array}$$

and so $R_d = \chi_R$ giving $\ulcorner R \urcorner_d = \ulcorner \chi_R \urcorner$ as required.

(iii) is immediate from (ii) since the counit is the mate of identity. □

5.2 Localic slice stability

We now embark on a proof that for any locale Y,

$$\mathsf{Loc}_{\mathrm{Sh}(Y)} \simeq \mathsf{Loc}/Y.$$

This result is of course well known, but its usual proof does require some understanding of constructions internal to a sheaf topos. Although we appear to be able to dispense with most of the topos theory, it does not appear possible to entirely dispense with sheaf theory as the following well-known observations are required:

Lemma 5.2.1. *For any locale Y,*

(a) $\mathrm{Sh}(Y) \simeq \mathsf{LH}/Y$,

(b) *the inclusion $\mathsf{LH}/Y \hookrightarrow \mathsf{Loc}/Y$ has a right adjoint. This right adjoint $(\_)_{d_Y}$ sends any X_f to the sheaf of sections, i.e. defined by*

$$(X_f)_{d_Y}(a) = \{s : a \to X \mid fs = a \hookrightarrow Y\}$$

for any open $a \hookrightarrow Y$ of Y, and

(c) *if $f : X \to Y$ is a map of locales then the pullback functor $f^* : \mathsf{LH}/Y \to \mathsf{LH}/X$ is cartesian and has a right adjoint, denoted by f_* say. As an action on sheaves, $f_*(F)(a) = F\Omega f(a)$.*

In part (c) we are following the notation whereby we use ΩX for the frame corresponding to a locale X and use $\Omega f : \Omega Y \to \Omega X$ for the frame homomorphism corresponding to a locale map $f : X \to Y$. Recall that Loc is defined to be $\mathsf{Fr}^{\mathrm{op}}$, the opposite of the category of frames, which itself can be defined as the category of algebras of the lower power set monad

$$\mathbb{D} \equiv (\mathcal{D}, \mathrm{Id} \xrightarrow{\downarrow} \mathcal{D}, \mathcal{D}\mathcal{D} \xrightarrow{\cup} \mathcal{D})$$

on $\wedge$-SLat.

Proof. Consult, for example, [3, Chapter VI]. In (c), the right adjoint is the direct image of the geometric morphism $f : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. $\square$

Note that for any locale map $f : X \to Y$ the adjunction $f^* \dashv f_* : \mathsf{LH}/X \leftrightarrows \mathsf{LH}/Y$ gives rise to an adjunction $f^* \dashv f_* : \wedge\text{-}\mathsf{SLat}_{\mathsf{LH}/X} \leftrightarrows \wedge\text{-}\mathsf{SLat}_{\mathsf{LH}/Y}$ since both f_* and f^* are cartesian and so preserve the relevant structure. Further note that, by exactly the same reasoning, in the situations where $i : \mathsf{Dis}_{\mathcal{C}} \hookrightarrow \mathcal{C}$ has a right adjoint $(\_)_d$, since i preserves finite products, there is an induced adjunction: $\wedge\text{-}\mathsf{SLat}_{\mathsf{Dis}_{\mathcal{C}}} \leftrightarrows \wedge\text{-}\mathsf{SLat}_{\mathcal{C}}$.

Next we characterize the situation $\mathcal{C} \simeq \mathsf{Loc}_{\mathsf{Dis}_{\mathcal{C}}}$; that is, we provide a categorical condition for when $\mathsf{Dis}_{\mathcal{C}}$ is a topos and $\mathcal{C}$ is the category of locales over $\mathsf{Dis}_{\mathcal{C}}$. Our main result for this section is essentially a proof that these conditions are slice stable.

Proposition 5.2.2. *$\mathcal{C}$ is a category satisfying the axioms. Then $\mathsf{Dis}_{\mathcal{C}}$ is a topos and $\mathcal{C} \simeq \mathsf{Loc}_{\mathsf{Dis}_{\mathcal{C}}}$ if and only if*

(i) *$\mathsf{Dis}_{\mathcal{C}} \hookrightarrow \mathcal{C}$ has a right adjoint, and*

(ii) *$\mathrm{Idl}(\_)^{\mathrm{op}} : \wedge\text{-}\mathsf{SLat}_{\mathsf{Dis}_{\mathcal{C}}} \to \mathcal{C}^{\mathrm{op}}$ has a monadic right adjoint.*

Proof. If $\mathsf{Dis}_{\mathcal{C}}$ is a topos and $\mathcal{C} \simeq \mathsf{Loc}_{\mathsf{Dis}_{\mathcal{C}}}$ then we have already constructed the right adjoint needed for (i). For (ii) send a locale X to ΩX.

In the other direction if $\mathsf{Dis}_{\mathcal{C}} \hookrightarrow \mathcal{C}$ has a right adjoint then by Proposition 5.1.3, $\mathsf{Dis}_{\mathcal{C}}$ is a topos. What remains is a verification that if $\mathrm{Idl}(\_)^{\mathrm{op}} : \wedge\text{-}\mathsf{SLat}_{\mathsf{Dis}_{\mathcal{C}}} \to \mathcal{C}^{\mathrm{op}}$ has a right adjoint (say denoted U) then the monad induced on $\wedge\text{-}\mathsf{SLat}_{\mathsf{Dis}_{\mathcal{C}}}$ is (naturally isomorphic to) the lower powerset monad $\mathbb{D}$ on $\wedge\text{-}\mathsf{SLat}_{\mathsf{Dis}_{\mathcal{C}}}$. The monadicity assumption that is part of condition (ii) then proves that $\mathcal{C} \simeq \mathsf{Loc}_{\mathsf{Dis}_{\mathcal{C}}}$.

Firstly, we check that for all meet semilattices A and B

$$\wedge\text{-}\mathsf{SLat}_{\mathsf{Dis}_{\mathcal{C}}}(A, U\mathrm{Idl}(B^{\mathrm{op}})) \cong \wedge\text{-}\mathsf{SLat}_{\mathsf{Dis}_{\mathcal{C}}}(A, \mathcal{D}B). \qquad (*)$$

Note that $\mathcal{D}B$ is the split of

$$PB \underset{\downarrow}{\overset{\mathrm{Id}}{\rightrightarrows}} PB,$$

where, of course, $PB = (\mathbb{S}^B)_d$. Since $\mathcal{D}B \hookrightarrow PB$ is a meet semilattice homomorphism it is easy to check that

$$\wedge\text{-}\mathsf{SLat}_{\mathsf{Dis}_{\mathcal{C}}}(A, \mathcal{D}B) \cong \{f : A \to PB \mid f\ \wedge\text{-}\mathsf{SLat}\text{ hom. and} \downarrow f = f\}.$$

Now

$$\wedge\text{-SLat}_{\mathsf{Dis}_{\mathcal{C}}}(A, U\mathrm{Idl}(B^{\mathrm{op}})) \cong \{n : A \to \mathbb{S}^B \mid n \ \wedge\text{-SLat hom. and } \alpha_{\geq_B} n = n\}$$

by Lemma 4.5.5. So to complete the proof of $(*)$ we must but check that

$$\downarrow = (\alpha_{\geq_B})_d \tag{a}$$

since we have commented already that the adjunction

$$i \dashv (\_)_d : \mathsf{Dis}_{\mathcal{C}} \rightleftarrows \mathcal{C}$$

gives rise to an adjunction

$$\wedge\text{-SLat}_{\mathsf{Dis}_{\mathcal{C}}} \rightleftarrows \wedge\text{-SLat}_{\mathcal{C}}.$$

For (a) it is sufficient to show for any monomorphism $R \hookrightarrow C \times B$ of $\mathsf{Dis}_{\mathcal{C}}$ that $\downarrow\ulcorner\chi_R\urcorner = (\alpha_{\geq_B})_d\ulcorner\chi_R\urcorner$. But by Lemma 5.1.5 the mate of $(\alpha_{\geq_B})_d\ulcorner\chi_R\urcorner$ under the adjunction $i \dashv (\_)_d$ is

$$C \xrightarrow{\ulcorner R \urcorner} \mathbb{S}^B \xrightarrow{\alpha_{\geq_B}} \mathbb{S}^B = C \xrightarrow{\ulcorner R;\geq_B \urcorner} \mathbb{S}^B.$$

Now $C \xrightarrow{\ulcorner\chi_R\urcorner} PB \xrightarrow{\downarrow} PB$ is equal to $C \xrightarrow{\ulcorner\chi_{R;\geq_B}\urcorner} PB$ since relational composition is calculated in $\mathsf{Dis}_{\mathcal{C}}$ and so uses the ; formulae developed. Therefore $\downarrow\ulcorner\chi_R\urcorner = (\alpha_{\geq_B})_d\ulcorner\chi_R\urcorner$ follows by another application of Lemma 5.1.5 and so (a) is established since we have argued with arbitrary R. Therefore we have established $(*)$ and so have established a natural isomorphism $\phi : \mathcal{D} \to U\mathrm{Idl}(\_)^{\mathrm{op}}$.

The remainder of the proof checks that the monad structure induced by the adjunction $\mathrm{Idl}(\_)^{\mathrm{op}} \dashv U$ is naturally isomorphic to $\mathbb{D}$. Note that by construction ϕ_B is the adjoint transpose of the opposite of a map $\mathrm{Idl}(B^{\mathrm{op}}) \to \mathrm{Idl}((\mathcal{D}B)^{\mathrm{op}})$ which itself corresponds to the membership relation $\in_B \hookrightarrow \mathcal{D}B \times B$. In other words, using the Notation 4.5.6, $\phi_B = \widetilde{n_{\in_B}^{\mathrm{op}}}$ where $\tilde{}$ denotes taking adjoint transpose via $\mathrm{Idl}(\_)^{\mathrm{op}} \dashv U$.

The unit of the monad induced by $\mathrm{Idl}(\_)^{\mathrm{op}} \dashv U$, evaluated at B, is given by taking the adjoint transpose of the identity on $\mathrm{Idl}(B^{\mathrm{op}})$. Given Lemma 4.5.5 we know that this corresponds to $B \xrightarrow{\ulcorner\geq_B\urcorner} \mathbb{S}^B$ and so to $\downarrow : B \to PB$ (Lemma 5.1.5 since $\downarrow = \ulcorner\chi_{\geq_B}\urcorner$). Therefore the unit of the monad induced by $\mathrm{Idl}(\_)^{\mathrm{op}} \dashv U$ maps to the unit of $\mathbb{D} \equiv (\mathcal{D}, \mathrm{Id} \xrightarrow{\downarrow} \mathcal{D}, \mathcal{D}\mathcal{D} \xrightarrow{\cup} \mathcal{D})$ via $(*)$.

To complete the proof we must check that the union operation $\mathcal{D}\mathcal{D} \xrightarrow{\cup} \mathcal{D}$ maps to $U\varepsilon_{\mathrm{Idl}(\_)^{\mathrm{op}}}$ via the natural isomorphism ϕ, where ε is the counit of the adjunction

$\mathrm{Idl}(\_)^{\mathrm{op}} \dashv U$. In other words, we must verify that the diagram

$$\begin{array}{ccc} \mathcal{D}\mathcal{D}B & \xrightarrow{U\mathrm{Idl}(\phi_B)\phi_{\mathcal{D}B}} & U\mathrm{Idl}(U\mathrm{Idl}(B^{\mathrm{op}}))^{\mathrm{op}} \\ \downarrow{\scriptstyle \cup} & & \downarrow{\scriptstyle U\varepsilon_{\mathrm{Idl}(B^{\mathrm{op}})}} \\ \mathcal{D}B & \xrightarrow{\phi_B} & U\mathrm{Idl}(B^{\mathrm{op}}) \end{array}$$

commutes. By taking adjoint transpose this amounts to checking that the diagram

$$\begin{array}{ccccc} \mathrm{Idl}(\mathcal{D}\mathcal{D}B^{\mathrm{op}}) & \xrightarrow{n^{\mathrm{op}}_{\in_{\mathcal{D}B}}} & \mathrm{Idl}(\mathcal{D}B^{\mathrm{op}}) & \xrightarrow{\mathrm{Idl}(\phi_B)} & \mathrm{Idl}(U\mathrm{Idl}(B^{\mathrm{op}}))^{\mathrm{op}} \\ \downarrow{\scriptstyle \mathrm{Idl}(\cup)} & & & & \downarrow{\scriptstyle \varepsilon_{\mathrm{Idl}(B^{\mathrm{op}})}} \\ \mathrm{Idl}(\mathcal{D}B^{\mathrm{op}}) & & \xrightarrow{n^{\mathrm{op}}_{\in_B}} & & \mathrm{Idl}(B^{\mathrm{op}}) \end{array}$$

commutes in $\mathcal{C}^{\mathrm{op}}$. Now $\varepsilon_{\mathrm{Idl}(B^{\mathrm{op}})}$ is the adjoint transpose of the identity $\mathrm{Id}_{U\mathrm{Idl}(B^{\mathrm{op}})}$ and since this identity factors as

$$U\mathrm{Idl}(B^{\mathrm{op}}) \xrightarrow{\phi_B^{-1}} \mathcal{D}B \xrightarrow{\phi_B} U\mathrm{Idl}(B^{\mathrm{op}})$$

(and $\phi_B = \widetilde{n^{\mathrm{op}}_{\in_B}}$) we get that $\varepsilon_{\mathrm{Idl}(B^{\mathrm{op}})} = n^{\mathrm{op}}_{\in_B}\mathrm{Idl}(\phi_B^{-1})$. So we are left checking that $n_{\in_{\mathcal{D}B}} n_{\in_B} = \mathrm{Idl}(\cup)^{\mathrm{op}} n_{\in_B}$. By Lemma 4.5.7 $n_{\in_{\mathcal{D}B}} n_{\in_B} = n_{\in_{\mathcal{D}B};\in_B}$ and by (iii) of Lemma 5.1.5 we get that the relation on $\mathcal{D}\mathcal{D}B \times \mathcal{D}B$ corresponding to $\mathrm{Idl}(\cup)^{\mathrm{op}}$ is (isomorphic to) the pullback of $\supseteq_{\mathcal{D}B} \hookrightarrow \mathcal{D}B \times \mathcal{D}B$ along $\cup \times \mathrm{Id}_{\mathcal{D}B}$. Since it is trivial to check that

$$\in_{\mathcal{D}B}; \in_B = [(\cup \times \mathrm{Id}_{\mathcal{D}B})^* \supseteq_{\mathcal{D}B}]; \in_B$$

we are done. □

Theorem 5.1. $\mathsf{Loc}_{\mathrm{Sh}(Y)} \simeq \mathsf{Loc}/Y$.

Proof. By part (b) of Lemma 5.2.1 and Proposition 5.1.3 we have that LH/Y is a topos since $\mathsf{LH}/Y = \mathsf{Dis}_{\mathsf{Loc}/Y}$ so by the previous proposition our proof reduces to proving that the functor $\mathrm{Idl}_Y(\_)^{\mathrm{op}} : \wedge\text{-}\mathsf{SLat}_{\mathsf{LH}/Y} \to \mathsf{Loc}^{\mathrm{op}}/Y$ has a monadic right adjoint (say denoted U_Y), and this is what will occupy the remainder of the proof.

If X_f is an object of Loc/Y define $U_Y(X_f)$ to be $f_*((\mathbb{S}_X)_{d_X})$. If $h : X'_{f'} \to X_f$ is a morphism of Loc/Y then let $U_Y(h) : f_*((\mathbb{S}_X)_{d_X}) \to f'_*((\mathbb{S}_{X'})_{d_{X'}})$ be the map $f_*(\widetilde{\chi_{h^*\top}})$ where $\widetilde{(\_)}$ is adjoint transpose via $h^* \dashv h_*$. The map $\top : 1_{X'} \to (\mathbb{S}_{X'})_{d_{X'}}$ is the universal monomorphism in LH/X' which exists since we have established that this category is a topos. The pullback (along h) of this monomorphism is a monomorphism and is therefore classified by a map to $(\mathbb{S}_X)_{d_X}$ and we are defining $U_Y(h)$ to be the f_* applied to the adjoint transpose of this classifying map.

We now check that this provides a right adjoint to $\mathrm{Idl}_Y(\_)^{\mathrm{op}}$. For any object A_g of $\wedge\text{-}\mathsf{SLat}_{\mathsf{LH}/Y}$, we have that

$$\begin{aligned}\wedge\text{-}\mathsf{SLat}_{\mathsf{LH}/Y}(A_g, f_*((\mathbb{S}_X)_{d_X})) &\cong \wedge\text{-}\mathsf{SLat}_{\mathsf{LH}/X}(f^*A_g, (\mathbb{S}_X)_{d_X})\\ &\cong \wedge\text{-}\mathsf{SLat}_{\mathsf{Loc}/X}(f^*A_g, \mathbb{S}_X).\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mathsf{Loc}/Y)^{\mathrm{op}}(\mathrm{Idl}_Y(A_g)^{\mathrm{op}}, X_f) &= (\mathsf{Loc}/Y)(X_f, \mathrm{Idl}_Y(A_g^{\mathrm{op}}))\\ &\cong (\mathsf{Loc}/X)(1, \mathrm{Idl}_X(f^*A_g^{\mathrm{op}}))\end{aligned}$$

since the Idl construction is stable under change of base. This verifies that $\mathrm{Idl}_Y(\_)^{\mathrm{op}} \dashv U_Y(X_f)$ since, by construction, the points of $\mathrm{Idl}_X(f^*A_g^{\mathrm{op}})$ are exactly the meet semilattice homomorphisms from f^*A_g to $\mathbb{S}_Y$. The naturality of the order isomorphisms involved is routine from our definition of U_Y on morphisms since (Lemma 5.1.5) the counit of the coreflection $(\_)_d$ classifies the universal monomorphism $\top$.

For completeness, we now check this naturality with respect to X_f. Say $t : A_g \to U_Y(X_f)$ and $\widehat{t} : X_f \to \mathrm{Idl}_Y(A_g^{\mathrm{op}})$ is its mate. We are given $h : X'_{f'} \to X_f$. Under the order isomorphisms between the homsets we have the following mappings:

$$A_g \xrightarrow{t} U_Y(X_f) \xrightarrow{U_Y(h)} U_Y(X'_{f'}),$$

$$A_g \xrightarrow{t} f_*((\mathbb{S}_X)_{d_X}) \xrightarrow{f_*(\widetilde{\chi_{h^*\top}})} f'_*((\mathbb{S}_{X'})_{d_{X'}}),$$

$$h^*(f^*A_g) \xrightarrow{h^*(\bar{t})} h^*((\mathbb{S}_X)_{d_X}) \xrightarrow{\chi_{h^*\top}} (\mathbb{S}_{X'})_{d_{X'}} \quad \bar{t} \text{ mate of } t \text{ via } f^* \!-\! ! f_*,$$

$$h^*(f^*A_g) \xrightarrow{h^*(\bar{t})} h^*((\mathbb{S}_X)_{d_X}) \xrightarrow{\chi_{h^*\top}} (\mathbb{S}_{X'})_{d_{X'}} \xrightarrow{\top'} \mathbb{S}_{X'},$$

where the last line is because $\top'$ is the counit of the adjunction $i_{X'} \dashv d_{X'}$, see Lemma 5.1.5 (iii). But

$$h^*((\mathbb{S}_X)_{d_X}) \xrightarrow{\chi_{h^*\top}} (\mathbb{S}_{X'})_{d_{X'}} \xrightarrow{\top'} \mathbb{S}_{X'}$$

is equal to $h^*\top$ since this is the open that it classifies and so we have that $U_Y(h)t$ maps to h^* applied to

$$f^*A_g \xrightarrow{\bar{t}} (\mathbb{S}_X)_{d_X} \xrightarrow{\top} \mathbb{S}_X.$$

On the other hand, $X'_{f'} \xrightarrow{h} X_f \xrightarrow{\widehat{t}} \mathrm{Idl}_Y(A_g^{\mathrm{op}})$ maps to the meet semilattice homomorphisms

$$A_g \longrightarrow \mathbb{S}_Y^{X_f} \xrightarrow{\mathbb{S}_Y^h} \mathbb{S}_Y^{X'_{f'}},$$

$$f^*A_g \xrightarrow{\top\bar{t}} \mathbb{S}_X \xrightarrow{\mathbb{S}_X^h} \mathbb{S}_X^{X'_h} \quad \text{change base to } X,$$

$$h^*f^*A_g \xrightarrow{h^*(\top\bar{t})} h^*(\mathbb{S}_X) \quad \text{change base to } X'.$$

This completes our check of naturality.

To verify that U_Y is monadic we check that (i) it has a left adjoint (clearly done already), (ii) it reflects isomorphisms and (iii) $(\mathsf{Loc}/Y)^{\mathrm{op}}$ has and U_Y preserves coequalizers of U_Y-split pairs (i.e. pairs $h_1, h_2 : X_f \to Z_g$ such that $U_Y(h_1)$, $U_Y(h_2)$ are part of a split coequalizer diagram in $\wedge$-$\mathsf{SLat}_{\mathsf{LH}/Y}$.)

To prove (ii) and (iii) we use an explicit description of the sheaf corresponding to $U_Y(X_f)$ and of the morphisms in Sh(Y) corresponding to $U_Y(h)$. Since $(\_)_{d_X}$ is the functor that takes the sheaf of sections, as a sheaf we have that $(\mathbb{S}_X)_{d_X}(a) \cong {\downarrow} a$ for every open $a \in \Omega X$. Therefore by the description of f_* given in part (c) of the lemma we have that the sheaf corresponding to $U_Y(X_f)$ is

$$U_Y(X_f)(b) \cong {\downarrow}\Omega f(b)$$

for every $b \in \Omega Y$.

As for $U_Y(h)$, note first that under the order isomorphism just established, for any $c \leq \Omega f(b)$, the point of $f_*(\mathbb{S}_X)_{d_X}$ corresponding to c is the adjoint transpose (under $f^* \dashv f_*$) of $\chi_c : f^*b \to (\mathbb{S}_X)_{d_X}$. The image of c under $U_Y(h)$ is therefore the adjoint transpose (under $(f')^* \dashv f'_*$) of

$$b \xrightarrow{\overline{\chi_c}} f_*((\mathbb{S}_X)_{d_X}) \xrightarrow{f_*(\widetilde{\chi_{h^*\top}})} f'_*((\mathbb{S}_{X'})_{d_{X'}})$$

which is $h^*(c)$. In other words, as a natural transformation (i.e. as a morphisms in Sh(Y)), $U_Y(h)(c) = \Omega h(c)$ for any $c \in U_Y(X_f)(b)$. Note that it is then immediate, by considering $b = 1$, that U_Y is conservative and so (ii) is checked.

For (iii) certainly $(\mathsf{Loc}/Y)^{\mathrm{op}}$ has such coequalizers as Loc is cartesian. Say

$$E_e \overset{i}{\hookrightarrow} X_f \underset{h_2}{\overset{h_1}{\rightrightarrows}} Z_g$$

is an equalizer diagram in Loc/Y such that there is a split coequalizer

$$U_Y(Z_g) \underset{U_Y(h_2)}{\overset{U_Y(h_1)}{\rightrightarrows}} U_Y(X_f) \overset{\gamma}{\twoheadrightarrow} F$$

in Sh(Y). We must check that $F \cong U_Y(E_e)$. Firstly, note that for any open $b \hookrightarrow Y$,

$$\downarrow\Omega g(b) \underset{\Omega h_2}{\overset{\Omega h_1}{\rightrightarrows}} \downarrow\Omega f(b) \xrightarrow{\Omega i} \downarrow\Omega e(b)$$

is a frame coequalizer since the pullback of the equalizer diagram

$$E \overset{i}{\hookrightarrow} X \underset{h_2}{\overset{h_1}{\rightrightarrows}} Z$$

along the open $\Omega g(b) \hookrightarrow Z$ is again an equalizer diagram in Loc. Now, for every open b of Y we have that

$$U_Y(Z_g)(b) \underset{U_Y(h_2)_b}{\overset{U_Y(h_1)_b}{\rightrightarrows}} U_Y(X_f)(b) \overset{\gamma_b}{\twoheadrightarrow} F(b) \qquad (*)$$

is a split coequalizer diagram in $\wedge$-SLat and since the category of frames is monadic over $\wedge$-SLat we have that $(*)$ is a coequalizer in the category of frames. But

$$U_Y(Z_g)(b) \underset{U_Y(h_2)_b}{\overset{U_Y(h_1)_b}{\rightrightarrows}} U_Y(X_f)(b)$$

is isomorphic to

$$\downarrow\Omega g(b) \underset{\Omega h_2}{\overset{\Omega h_1}{\rightrightarrows}} \downarrow\Omega f(b)$$

and so $F(b) \cong \downarrow\Omega e(b) \cong U_Y(E_e)(b)$ for every b. Hence $F \cong U_Y(E_e)$ as required. This completes the proof of (iii) and so shows that U_Y is monadic as required. □

6 Concluding comments

It should be clear that the categorical axioms offered here are not as elegant as could be hoped for. There is no claim in this work that the axioms represent the best way of developing locale theory based purely on categorical assumptions. For example, the interdependencies between the axioms have not been explored and so some axioms may be redundant. The point of the work is rather that (a) the axioms 'do the job' in the sense that they allow us to recover the well-known constructions and results, (b) the theory is order dual and so simultaneously covers both the theory of proper maps and the theory of open maps and (c) the axioms are slice stable. This paper has focused on (c) and we have covered the following aspects:

(i) the axiomatic framework is slice stable,

(ii) the fundamental theorem of topos theory can be recovered, and

(iii) the logical slice stability of locales can be proved using only some basic assumptions from sheaf theory.

Along the way we have offered a new categorical proof of Vicker's result that $P_L(A) \cong \mathbb{S}^A$ for discrete A and have developed, axiomatically, some well-known properties of the ideal completion locale construction. We end with Table 1 of the axioms together with some technical comments on them.

7 Appendix

7.1 Product as join semilattice tensor

The following axiom is used in [10]. It asserts that tensor exists in $\mathcal{C}_{P_L}^{\mathrm{op}}$ and is given by product in $\mathcal{C}$. In the case that $\mathcal{C} = \mathsf{Loc}$ this is well known since suplattice tensor provides a description of frame coproduct (i.e. of locale product).

Axiom 9. For any objects X and Y of $\mathcal{C}$, the map $\otimes : \mathbb{S}^X \times \mathbb{S}^Y \to \mathbb{S}^{X \times Y}$ defined by $\sqcap_{\mathbb{S}^{X\times Y}}(\mathbb{S}^{\pi_1} \times \mathbb{S}^{\pi_2})$ is universal $\sqcup$-bilinear.

It is not immediate that this axiom extends beyond the binary case for which it is given. Though we would expect, for example, that the map $\otimes : \mathbb{S}^X \times \mathbb{S}^Y \times \mathbb{S}^Z \to \mathbb{S}^{X\times Y\times Z}$ is necessarily universal $\sqcup$-trilinear given the axiom, the usual proof of this fact does not work because the category $\mathcal{C}_{P_L}^{\mathrm{op}}$ is not cartesian closed. However, given Axiom 9, n-ary tensor does exist for $n \geq 2$, but the proof of this seems to require a slightly involved lemma which we now provide.

Lemma 7.1.1. *Given Axiom 9 and any objects X, Y and Z of $\mathcal{C}$, the map $\otimes$: $\mathbb{S}^X \times \mathbb{S}^Y \times \mathbb{S}^Z \to \mathbb{S}^{X\times Y\times Z}$ defined by $\sqcap_{\mathbb{S}^{X\times Y\times Z}}(\mathrm{Id} \times \sqcap_{\mathbb{S}^{X\times Y\times Z}})(\mathbb{S}^{\pi_1} \times \mathbb{S}^{\pi_2} \times \mathbb{S}^{\pi_3})$ is universal $\sqcup$-trilinear.*

Proof. Certainly $\otimes : \mathbb{S}^X \times \mathbb{S}^Y \times \mathbb{S}^Z \to \mathbb{S}^{X\times Y\times Z}$ is join trilinear. We must show that given any join trilinear $\psi : \mathbb{S}^X \times \mathbb{S}^Y \times \mathbb{S}^Z \to \mathbb{S}^R$ that there exists a unique join semilattice homomorphism $\delta : \mathbb{S}^{X\times Y\times Z} \to \mathbb{S}^R$ such that $\delta\otimes = \psi$.

Axiom	Description	Comment
1	Order enriched cartesian	
2	Stable distributive	Necessity of slice stable phrasing open
3	Sierpiński exists	Not canonical
4	Equalizers give rise to coequalizers in presheaves over $\mathcal{C}$	Axiomatic version of the double coverage theorem. Equivalent to $\mathbb{P}$ preserving coreflexive equalizers in the presence of Axioms 7 and 9
5	$\mathbb{S}^{(\_)}$ reflects isomorphisms	
5′	Distributive lattice homs. $\mathbb{S}^Y \to \mathbb{S}^X$ are of the form $\mathbb{S}^f$ for unique $f : X \to Y$	Strengthening of Axiom 5
6	$\mathbb{S}^{(\_)}$ takes regular epimorphisms to monomorphisms	In locale theory $\mathbb{S}^q$ is a regular monomorphism for regular epimorphism q but this strengthening is not required in application.
7	Double exponentiation over $\mathbb{S}$ exists	Existence necessary for the other two power objects and the ideal completion constructions
8	Power monads are KZ/coKZ	Implied by Cauchy completeness of Kleisli categories. Implies that idempotent inflationary/deflationary Kleisli morphisms split. This splitting (on the lower side of the duality) gives ideal completion.
9	Product is given by tensor	Can be used to prove pullback stability results. Open whether necessary as an additional axiom.

Table 1

Given any $c : W \times Z \to \mathbb{S}$ and W' an object of $\mathcal{C}$, define

$$\phi^c_{W'} : \mathcal{C}(W' \times X, \mathbb{S}) \times \mathcal{C}(W' \times Y, \mathbb{S}) \to \mathcal{C}(W' \times W \times R, \mathbb{S}),$$
$$(a, b) \mapsto \psi_{W' \times W}(a\pi_{13}, b\pi_{13}, c\pi_{23}).$$

It is routine to verify that this is natural in W' and further it is routine to verify that the natural transformation $\phi^c : \mathbb{S}^X \times \mathbb{S}^Y \to \mathbb{S}^{W \times R}$ so defined is join bilinear. By Axiom 9 there exists a unique join semilattice homomorphism $\gamma^c : \mathbb{S}^{X \times Y} \to \mathbb{S}^{W \times R}$ such that $\gamma^c \otimes = \phi^c$. By uniqueness note that for any $c_1, c_2 : W \times Z \to \mathbb{S}$, $\gamma^{c_1 \sqcup c_2} = \gamma^{c_1} \sqcup \gamma^{c_2}$ and $\gamma^0 = 0$. Also by uniqueness γ^c is natural in c; if $g : W_2 \to W_1$ and $c : W_1 \times Z \to \mathbb{S}$ then for arbitrary W', $\gamma^{c(g \times \mathrm{Id}_Z)}_{W'}(a \otimes b)$ equals

$$\begin{aligned}
&\psi_{W' \times W_2}(a\pi_{13}, b\pi_{13}, c(g \times \mathrm{Id}_Z)\pi_{23}) \\
&\quad = \psi_{W' \times W_2}(a\pi_{13}(\mathrm{Id}_{W'} \times g \times \mathrm{Id}_X), b\pi_{13}(\mathrm{Id}_{W'} \times g \times \mathrm{Id}_Y), \\
&\qquad c\pi_{23}(\mathrm{Id}_{W'} \times g \times \mathrm{Id}_Z)) \\
&\quad = (\mathrm{Id}_{W'} \times g \times \mathrm{Id}_R)\psi_{W' \times W_1}(a\pi_{13}, b\pi_{13}, c\pi_{23}) \\
&\quad = (\mathrm{Id}_{W'} \times g \times \mathrm{Id}_R)\gamma^c_{W'}(a \otimes b)
\end{aligned}$$

which implies that $\mathbb{S}^{g \times \mathrm{Id}_R}\gamma^c \otimes = \gamma^{c(g \times \mathrm{Id}_Z)} \otimes$ and so $\mathbb{S}^{g \times \mathrm{Id}_R}\gamma^c = \gamma^{c(g \times \mathrm{Id}_Z)}$ by uniqueness as $\mathbb{S}^{g \times \mathrm{Id}_R}$ is a join semilattice homomorphism. This verifies our claim that γ^c is natural in c.

We now use γ^c and its properties to define a join bilinear map $\mathbb{S}^{X \times Y} \times \mathbb{S}^Z \to \mathbb{S}^R$ which will therefore extend to our required map $\delta : \mathbb{S}^{X \times Y \times Z} \to \mathbb{S}^R$ by the application of Axiom 9. For each object W of $\mathcal{C}$ define a map

$$\rho_W : \mathcal{C}(W \times X \times Y, \mathbb{S}) \times \mathcal{C}(W \times Z, \mathbb{S}) \to \mathcal{C}(W \times R, \mathbb{S}),$$
$$(I, c) \mapsto \gamma^c_W(I) \circ (\Delta_W \times \mathrm{Id}_R).$$

This is natural in W (use the naturality of γ^c in c which we have just established). The natural transformation $\rho : \mathbb{S}^{X \times Y} \times \mathbb{S}^Z \to \mathbb{S}^R$ so defined is join bilinear since γ^c is a join semilattice homomorphism and $\gamma^{c_1 \sqcup c_2} = \gamma^{c_1} \sqcup \gamma^{c_2}$ (and $\gamma^0 = 0$). By Axiom 9 there exists a join semilattice homomorphism $\delta : \mathbb{S}^{X \times Y \times Z} \to \mathbb{S}^R$ such that $\delta \otimes^{X \times Y, Z} = \rho$ where $\otimes^{X \times Y, Z} : \mathbb{S}^{X \times Y} \times \mathbb{S}^Z \to \mathbb{S}^{X \times Y \times Z}$ is the universal join bilinear tensor. The join trilinear tensor $\otimes : \mathbb{S}^X \times \mathbb{S}^Y \times \mathbb{S}^Z \to \mathbb{S}^{X \times Y \times Z}$ factors as

$$\otimes : \mathbb{S}^X \times \mathbb{S}^Y \times \mathbb{S}^Z \xrightarrow{\otimes \times \mathrm{Id}} \mathbb{S}^{X \times Y} \times \mathbb{S}^Z \xrightarrow{\otimes^{X \times Y, Z}} \mathbb{S}^{X \times Y \times Z}$$

and so to complete the existence part of this proof we must just check that

$$\begin{array}{ccc} \mathbb{S}^X \times \mathbb{S}^Y \times \mathbb{S}^Z & \xrightarrow{\otimes \times \mathrm{Id}} & \mathbb{S}^{X \times Y} \times \mathbb{S}^Z \\ & \underset{\psi}{\searrow} & \downarrow \rho \\ & & \mathbb{S}^R \end{array}$$

commutes. But for each W we have

$$\begin{aligned} \rho_W(a \otimes b, c) &= \gamma^c_W(a \otimes b) \circ (\Delta_W \times \mathrm{Id}_R) \\ &= \psi_{W \times W}(a\pi_{13}, b\pi_{13}, c\pi_{23}) \circ (\Delta_W \times \mathrm{Id}_R) \\ &= \psi_W(a\pi_{13}(\Delta_W \times \mathrm{Id}_X), b\pi_{13}(\Delta_W \times \mathrm{Id}_Y), c\pi_{23}(\Delta_W \times \mathrm{Id}_Z)) \\ &= \psi_W(a, b, c) \end{aligned}$$

by the naturality of ψ and so existence is established.

For uniqueness say $\delta' : \mathbb{S}^{X \times Y \times Z} \to \mathbb{S}^R$ is also a join semilattice homomorphism with $\delta' \otimes = \psi$. For each $c : W \times Z \to \mathbb{S}$ define $(\gamma')^c : \mathbb{S}^{X \times Y} \to \mathbb{S}^{W \times R}$ by

$$\begin{aligned} (\gamma')^c_{W'} : \mathcal{C}(W' \times X \times Y, \mathbb{S}) &\to \mathcal{C}(W' \times W \times R, \mathbb{S}), \\ I &\mapsto \delta'_{W' \times W} \otimes^{X \times Y, Z}_{W' \times W} (I\pi_{134}, c\pi_{23}). \end{aligned}$$

It is routine to verify that this is natural in W'. The morphism $(\gamma')^c$ so defined is a join semilattice homomorphism since δ' is.

By naturality,

$$\delta'_W \otimes^{X \times Y, Z}_W (I, c) = (\gamma')^c_W(I) \circ (\Delta_W \times \mathrm{Id}_R),$$

so the proof will be complete provided we can show that $(\gamma')^c = \gamma^c$, since $\delta_W \otimes^{X \times Y, Z}_W (I, c) = \gamma^c_W(I) \circ (\Delta_W \times \mathrm{Id}_R)$ by construction. To show that indeed $(\gamma')^c = \gamma^c$ it is sufficient to check that $(\gamma')^c \otimes = \gamma^c \otimes$, i.e. that

$$\delta'_{W' \times W} \otimes^{X \times Y, Z}_{W' \times W} ((a \otimes b)\pi_{134}, c\pi_{23}) = \psi_{W' \times W}(a\pi_{13}, b\pi_{13}, c\pi_{23})$$

for all $a : W' \times X \to \mathbb{S}$ and $b : W' \times Y \to \mathbb{S}$. Since

$$\delta'_{W' \times W} \otimes^{X \times Y, Z}_{W' \times W} (\otimes_{W' \times W} \times \mathrm{Id}) = \psi_{W' \times W},$$

by assumption (as the trilinear tensor factors as $\otimes^{X \times Y, Z}_{W' \times W} (\otimes_{W' \times W} \times \mathrm{Id})$), we therefore are left checking that $\otimes_{W' \times W}(a\pi_{13}, b\pi_{13}) = (a \otimes b)\pi_{134}$. This is a routine unwinding of the definition of the natural transformation $\otimes : \mathbb{S}^X \times \mathbb{S}^Y \to \mathbb{S}^{X \times Y}$;

$$\begin{aligned} \otimes_{W' \times W}(a\pi_{13}, b\pi_{13}) &= \sqcap_{\mathbb{S}}(a\pi_{123}, b\pi_{124}) \\ &= \sqcap_{\mathbb{S}}(a\pi_{12}, b\pi_{13})\pi_{134} \\ &= \otimes_{W'}(a, b)\pi_{134} = (a \otimes b)\pi_{134}. \end{aligned}$$

□

Proposition 7.1.2. *Assuming Axioms* 1–4, *Axiom* 9 *is slice stable.*

Proof. If A, B and C are objects of $\mathcal{C}/Y$ then we must check that any join bilinear $\psi : \mathbb{S}_Y^A \times \mathbb{S}_Y^B \to \mathbb{S}_Y^C$ extends uniquely, via $\otimes : \mathbb{S}_Y^A \times \mathbb{S}_Y^B \to \mathbb{S}_Y^{A\times B}$, to a join semilattice homomorphism.

Firstly, note that if $A = X_Y$ and $B = X'_Y$ for some objects X and X' of $\mathcal{C}$ then the result holds by change of base (from Y to 1). But for arbitrary X_f an object of $\mathcal{C}/Y$, there exists an equalizer diagram

$$X_f \xhookrightarrow{(\mathrm{Id},f)} X_Y \underset{\Delta_Y \pi_2}{\overset{f\times \mathrm{Id}}{\rightrightarrows}} Y_Y$$

in $\mathcal{C}/Y$. The proof will therefore be complete provided we can be show that the axiom is stable under taking equalizers. For this it is sufficient to check, given an equalizer diagram

$$E \xhookrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

in $\mathcal{C}$ and object Z of $\mathcal{C}$ with X, Y and Z all satisfying the axiom, that E and Z also satisfy the axiom. Say that we are given join bilinear $\psi : \mathbb{S}^E \times \mathbb{S}^Z \to \mathbb{S}^W$ for some arbitrary object W. By the lower coverage theorem join semilattice homomorphisms $\delta : \mathbb{S}^{E\times Z} \to \mathbb{S}^W$ are in order isomorphism with join semilattice homomorphisms $\beta : \mathbb{S}^{X\times Z} \to \mathbb{S}^W$ such that

$$\beta \sqcap_{\mathbb{S}^{X\times Z}} (\mathrm{Id} \times \mathbb{S}^{f\times \mathrm{Id}_Z}) = \beta \sqcap_{\mathbb{S}^{X\times Z}} (\mathrm{Id} \times \mathbb{S}^{g\times \mathrm{Id}_Z}) \qquad (*)$$

since

$$E \times Z \xhookrightarrow{e\times \mathrm{Id}_Z} X \times Z \underset{g\times \mathrm{Id}_Z}{\overset{f\times \mathrm{Id}_Z}{\rightrightarrows}} Y \times Z$$

is an equalizer diagram. In fact, condition $(*)$ is equivalent to β composing equally with

$$\mathbb{S}^X \times \mathbb{S}^Z \times \mathbb{S}^Y \times \mathbb{S}^Z \xrightarrow{\otimes} \mathbb{S}^{X\times Z\times Y\times Z} \xrightarrow{\mathbb{S}^{(\pi_1,\pi_2,f\pi_1,\pi_2)}} \mathbb{S}^{X\times Z} \qquad \text{(a)}$$

and

$$\mathbb{S}^X \times \mathbb{S}^Z \times \mathbb{S}^Y \times \mathbb{S}^Z \xrightarrow{\otimes} \mathbb{S}^{X\times Z\times Y\times Z} \xrightarrow{\mathbb{S}^{(\pi_1,\pi_2,g\pi_1,\pi_2)}} \mathbb{S}^{X\times Z} \qquad \text{(b)}$$

by the previous lemma where the $\otimes s$ are 4-ary tensors. But (a) factors as

$$\mathbb{S}^X \times \mathbb{S}^Z \times \mathbb{S}^Y \times \mathbb{S}^Z \xrightarrow{\sqcap_{\mathbb{S}^Y \times \mathbb{S}^Z} (\mathrm{Id}\times\mathrm{Id}\times\mathbb{S}^f\times\mathrm{Id})} \mathbb{S}^X \times \mathbb{S}^Z \xrightarrow{\otimes} \mathbb{S}^{X\times Z} \qquad \text{(a}')$$

and (b) factors as

$$\mathbb{S}^X \times \mathbb{S}^Z \times \mathbb{S}^Y \times \mathbb{S}^Z \xrightarrow{\sqcap_{\mathbb{S}^Y \times \mathbb{S}^Z}(\mathrm{Id}\times\mathrm{Id}\times\mathbb{S}^g\times\mathrm{Id})} \mathbb{S}^X \times \mathbb{S}^Z \xrightarrow{\otimes} \mathbb{S}^{X\times Z} \qquad (\mathrm{b}')$$

and so since $\mathbb{S}^e \times \mathrm{Id} : \mathbb{S}^X \times \mathbb{S}^Z \to \mathbb{S}^E \times \mathbb{S}^Z$ composes equally with the first factor of (a′) and the first factor of (b′) we have that the extension

$$\overline{\psi(\mathbb{S}^e \times \mathrm{Id})} : \mathbb{S}^{X\times Z} \to \mathbb{S}^W$$

of the join bilinear map $\psi(\mathbb{S}^e \times \mathrm{Id}) : \mathbb{S}^X \times \mathbb{S}^Z \to \mathbb{S}^W$ must factor via

$$\mathbb{S}^{X\times Z} \xrightarrow{\mathbb{S}^{e\times \mathrm{Id}_Z}} \mathbb{S}^{E\times Z},$$

and so there exists a join semilattice homomorphism $\delta : \mathbb{S}^{E\times Z} \to \mathbb{S}^W$ with the property that $\delta\mathbb{S}^{e\times\mathrm{Id}_Z} = \overline{\psi(\mathbb{S}^e \times \mathrm{Id})}$. Certainly the diagram

$$\begin{array}{ccc} \mathbb{S}^{X\times Z} & \xrightarrow{\mathbb{S}^{e\times\mathrm{Id}_Z}} & \mathbb{S}^{E\times Z} \\ \uparrow{\scriptstyle \otimes^{X,Z}} & & \uparrow{\scriptstyle \otimes^{E,Z}} \\ \mathbb{S}^X \times \mathbb{S}^Z & \xrightarrow{\mathbb{S}^e\times\mathrm{Id}} & \mathbb{S}^E \times \mathbb{S}^Z \end{array}$$

commutes and so $\delta \otimes^{E,Z} (\mathbb{S}^e \times \mathrm{Id}) = \psi(\mathbb{S}^e \times \mathrm{Id})$ and since, as in the proof of the lower coverage theorem, $\mathbb{S}^e \times \mathrm{Id}$ is a an epimorphism this shows that $\delta\otimes^{E,Z} = \psi$ as required. The proof is therefore completed with the observation that uniqueness is trivial since (i) $\mathbb{S}^{e\times\mathrm{Id}_Z}$ is an epimorphism and (ii), by assumption, $\otimes^{X,Z}$ is universal. □

All of the results of [10] are therefore slice stable. In particular, applying those techniques to the exposition on subgroupoids contained within [4, Theorem 5.3.1], we conjecture that it can be shown axiomatically that (i) any subgroupoid $\mathbb{H}$ of $\mathbb{G}$ with open domain and codomain maps is weakly closed over the object G_0 of $\mathbb{G}$ and (ii) any subgroupoid $\mathbb{H}$ of $\mathbb{G}$ with proper domain and codomain maps is fitted over the object G_0 of $\mathbb{G}$.

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Received May 11, 2010.

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