The patch construction is dual to algebraic dcpo representation

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Abstract

Using the parallel between the preframe and the suplattice approach to locale theory it is shown that the patch construction, as an action on topologies, is the same thing as the process of recovering a discrete poset from its algebraic dcpo (ideal completion).

1 Introduction

There is a fundamental observation in the theory of partially ordered sets that any partially ordered set (poset) can be recovered from its ideal completion. An isomorphic copy of any poset can be found as the subset of compact elements within its own ideal completion. Turning to a seemingly completely different area, given a compact Hausdorff poset $(X, \leq)$, with $\leq$ therefore necessarily a closed subset, we can form a new topology by looking at the set of complements of upper closed and topologically closed subsets of X. Then, via the patch construction (4.5 [J82]), the original compact Hausdorff poset can be recovered. The purpose of this paper is to show that both these results can be derived using the same abstract argument.

Both these motivational observations can be expressed as categorical equivalences. The former is the statement that the algebraic directed complete partial orders (algebraic dcpos), with compact element preserving dcpo homomorphisms, is equivalent to the category of partially ordered sets and monotone maps. The latter is the statement that the sober stably locally compact spaces (with perfect maps) is equivalent to the category compact Hausdorff posets with monotone maps. The truth of the former equivalence is central to lattice theory being a key aspect of the theory of algebraic lattices which emerged in universal algebra through the study of congruences (say, [BF48]). This equivalence also allows for the information system approach to denotational semantics to develop and so impacts on theoretical computer science, [S82]. By placing the discrete topology on a poset and the Scott topology on its algebraic dcpo, the equivalence can be viewed topologically. As for the latter equivalence between compact Hausdorff posets and sober stably locally compact spaces, the germ of this idea can be found by combining Priestley's duality, [P70], with Stone's representation theorem for distributive lattices, [S37]. The equivalence first appears in [G80] though see [H84] for more detail. The equivalence is an important aspect of topological lattice theory.

In fact both equivalences work localically; that is, they can be stated and proved relative to the category of locales (as opposed to the category of topological spaces). The locale theory approach to topology takes as its primitive complete Heyting algebras, viewing these as topologies and thereby sidestepping the need to define an ambient set of points, [J82]. The former equivalence, localically, exhibits the algebraic dcpos (here called ideal completion locales, as objects within the category of locales) equivalent to the discrete localic posets. This is an easy theorem from the definitions in locale theory. The latter example exhibits the stably locally compact locales (what else?) equivalent to the category of ordered compact Hausdorff locales. The advantage of the localic statement of the results is that they are valid in an arbitrary topos and so are logically more general than the usual topological results. The motivational results can be recovered, the former trivially and the latter by appealing to the spatiality of compact Hausdorff locales (using the prime ideal theorem). The second pay-off of moving to locales, which we hope to exhibit in this paper, is that under an order duality ([T96], [T06], [T05]) compact Hausdorff corresponds to discrete and that further under this duality the latter and former equivalences are the same result.

The first section covers the basic definitions of locale theory, and sets up the definitions of the various lattice theoretic maps that we will be needed (suplattice and preframe homomorphisms). The second section provides a representation theorem for both open relations on discrete locales and closed relations on compact Hausdorff locales. The representation is in terms of suplattice and preframe homomorphisms respectively. Both representations map relational composition to function composition. The next section shows that by splitting idempotent suplattice (and preframe) homomorphisms between frames what we informal name "up-set" locales arise and that, by construction, these have a Lawson style duality. This section is a specialization of the previous section and ends with some technical observations on how the evaluation map on the duality corresponds to the ordering on the poset of which an "up-set" locale is constructed. Section 4 contains the key technical result which shows how to recover any poset from its "up-set" locale. The proof reasons with suplattice homomorphisms for discrete posets and follows identical reasoning on preframe homomorphisms for ordered compact Hausdorff locales. Section 5 then clarifies that these "up-set" locales are in fact well known and correspond to the ideal completion locales (i.e. locales whose posets of points are ideal completions) and the stably locally compact locales. We can then state the key technical result as our main theorem:

Theorem 1 *(i) There is a bijection between discrete locales and ideal completion locales,*

(ii) There is a bijection between ordered compact Hausdorff locales and stably locally compact locales.

Let us briefly outline the central argument. Given a partially ordered set

$(X, \leq)$, the power set of X embeds into $\mathcal{U}(X^{op} \times X)$, where $\mathcal{U}$ denotes taking the set of upper closed subsets. Any subset I of X maps to

$$R_I \equiv \bigcup_{i \in I} \downarrow i \times \uparrow i.$$

Certainly R_I enjoys

$$R_I = \leq ; (R_I \cap \Delta); \leq$$

where Δ is the diagonal and ; denotes relational composition and indeed any R which enjoys this property must be of the form R_I for a unique subset I. Thus PX, the power set of X, can be recovered from $\mathcal{U}(X^{op} \times X)$ by looking at the fixed points of some idempotent endomorphism on $\mathcal{U}(X^{op} \times X)$. Moreover this endomorphism can be defined purely in terms of relational composition (and, we shall see, can be defined without reference to the relation $\leq$). Thus we can obtain the opens of the discrete space X by an argument involving relational composition. But, the category of compact Hausdorff spaces is regular (for example, by Manes' theorem [M67], [M69]) and so relational composition is also available. For any compact Hausdorff poset $(X, \leq)$ one can form a space whose opens are the complements of upper closed and topologically closed subsets; this is analogous to $\mathcal{U}(\_)$. Provided we can demonstrate that morphisms on this set of opens correspond to relations on X (and that function composition corresponds to relational composition) the same argument is available and the opens of any compact Hausdorff poset can be recovered. It then becomes routine to verify that this is the known patch construction. The main insight for the paper is therefore that the patch construction, as an action on topologies, can be interpreted spatially and is the compact Hausdorff analogue of the process of backing out PX from $\mathcal{U}(X^{op} \times X)$.

That there might be a parallel between the patch construction and information system representation is first explicit in [T96], under the guidance of Vickers. What is omitted there is any sense that the techniques needed for localic patch are the same as the techniques needed for backing out a localic discrete poset from its algebraic dcpo; but with the result presented here this is now available. With the proviso that we must define, abstractly, a stably locally compact locale to be the 'up-set' of an ordered compact Hausdorff poset, the techniques of [Ta00] or [T05] can be applied to ensure that this parallel is a formal categorical order duality.

2 Locale Theory Background

In this section we recall the basic definitions of locale theory. A *frame* is a complete lattice for which arbitrary joins distribute over finite meets, the motivating example being the set of opens of a topological space. A frame homomorphism is required to preserve arbitrary joins and finite meets, and so the category **Fr** is defined. The category of locales, **Loc**, is by definition the opposite of the category of frames (same objects, but formally reversed arrows). We follow a notation whereby a frame is always denoted ΩX, and X is called the *corresponding locale*. Frame homomorphisms are written $\Omega f : \Omega Y \to \Omega X$, so $f : X \to Y$ is the corresponding locale map. Consult [J82] for background on basic lattice theory and the theory of locales. The initial frame is the power set of the singleton set $\{*\}$ and we write $\Omega \equiv P\{*\}$, so $\Omega 1$, the frame of the terminal locale, is written Ω.

Weaker than frame homomorphisms we have preframe homomorphisms (**PreFr**), required to preserve directed joins and finite meets, and suplattice homomorphisms (**sup**) required to preserve arbitrary joins. These weaker notions are central to locale theory since locale product can in fact be described using either suplattice tensor of preframe tensor. For any locales X, Y

$$\Omega X \otimes_{\mathbf{PreFr}} \Omega Y \cong \Omega(X \times Y) \cong \Omega X \otimes_{\mathbf{sup}} \Omega Y. \qquad (*)$$

That such tensors can be defined as universal objects is shown for suplattice tensor in [JT84] and for preframe tensor in [JV91]. Taking $Y = 1$ we see that Ω is the unit both for preframe tensor and suplattice tensor; that is,

$$\Omega X \otimes_{\mathbf{PreFr}} \Omega \cong \Omega X \cong \Omega X \otimes_{\mathbf{sup}} \Omega$$

and we shall pass through both order isomorphisms without notation in what follows. If $\Delta : X \hookrightarrow X \times X$ is the diagonal then for any opens $a, b \in \Omega X$ we have both

$$\Omega\Delta(a \otimes b) = a \wedge b$$

and

$$\Omega\Delta(a \odot b) = a \vee b$$

where $\otimes$ is suplattice tensor and $\odot$ is preframe tensor. To see the latter from the former note that the order isomorphism $(*)$ relates $\otimes$ to $\odot$ via

$$a \odot b = a \otimes 1 \vee 1 \otimes b.$$

A locale X is said to be *open* provided the unique frame homomorphism $\Omega !^X : \Omega \longrightarrow \Omega X$ has a left adjoint. Such a left adjoint is necessarily a suplattice homomorphism. If X is a set then PX is the frame of opens of an open locale since $\exists_{!^X} : PX \to \Omega$, defined by $\exists_{!^X}(I) = 1$ iff $\exists i \in I$, is the left adjoint. Extending the notation, if $f : X \to Y$ is a locale map then

$$\exists_f : \Omega X \to \Omega Y$$

denotes the left adjoint to $\Omega f : \Omega Y \to \Omega X$ when it exists. Say, as an example, X is an open locale and Y, W are two other arbitrary locales then for

$$\pi_{13} : Y \times X \times W \to Y \times W$$

we have

$$\exists_{\pi_{13}} = Id_{\Omega Y} \otimes \exists_{!^X} \otimes Id_{\Omega W}$$

where we use Id to denote the identity morphisms. This can be derived by appealing to uniqueness of left adjoints. In what follows we will also need,

$$\exists_{\pi_{13}} = Id_{\Omega W} \otimes \exists_{\pi_2}$$

where $\pi_2 : X \times W \to W$. This again follows by uniqueness of left adjoints.

By exchanging suplattice homomorphism with preframe homomorphism and 'left adjoint' with 'right adjoint', the same analysis exists for compact locales. A locale X is compact if the right adjoint to $\Omega !^X : \Omega \longrightarrow \Omega X$ is a preframe homomorphism. If (X, τ) is a topological space then it is compact if and only if τ is the frame of opens of a compact locale. To see this construct the map $\forall_{!X} : \tau \to \Omega$ given by $\forall_{!X}(U) = 1$ iff $X = U$; it is a preframe homomorphism if and only if (X, τ) is a compact topological space. The general notation is

$$\forall_f : \Omega X \to \Omega Y$$

for the right adjoint of any frame homomorphism $\Omega f : \Omega Y \to \Omega X$. Such a right adjoint always exists; but we will only be interested in it when it is a preframe homomorphism. As in the suplattice analysis we have

$$\forall_{\pi_{13}} = Id_{\Omega Y} \odot \forall_{!X} \odot Id_{\Omega W}$$

for $\pi_{13} : Y \times X \times W \to Y \times W$ with X compact and Y, W arbitrary. As an action on preframe generators this sends $b \odot a \odot c$ to $b \odot c \vee \bigvee_{1 \leq a} 1_{\Omega Y \otimes_{\mathbf{PreFr}} \Omega W}$. Also,

$$\forall_{\pi_{13}} = Id_{\Omega W} \odot \forall_{\pi_2}$$

where $\pi_2 : X \times W \to W$.

A locale map $i : X_0 \to X$ is a *sublocale* (or X_0 is a sublocale) if Ωi is a frame surjection. Sublocales of X can be ordered in the obvious manner by saying that $i : X_0 \hookrightarrow X$ is less than or equal to $i' : X_0' \hookrightarrow X$ if and only if i factors via i'. For any $a \in \Omega X$ there are two frame surjections:

$$\begin{array}{rcl} \Omega i_a & : & \Omega X \to \downarrow a \\ b & \longmapsto & a \wedge b \end{array}$$

and

$$\begin{array}{rcl} \Omega i_{\neg a} & : & \Omega X \to \uparrow a \\ b & \longmapsto & a \vee b \end{array}$$

The corresponding sublocales are denoted $a \hookrightarrow X$ and $\neg a \hookrightarrow X$ respectively and are known as *open* and *closed* sublocales. It is important to note that, in this context, $\neg a$ is not the Heyting negation of a but is notation for the sublocale that is the closed complement of the open sublocale $a \hookrightarrow X$. $\exists_{i_a}$exists for open i_a and $\forall_{i_{\neg a}}$ is a preframe homomorphism for closed $i_{\neg a}$; notice that $a = \exists_{i_a}(1)$ and $a = \forall_{i_{\neg a}}(0)$. Using $OSub(X)$ (respectively $CSub(X)$) to denote the poset of open sublocales (respectively closed sublocales) it can be checked that there are order isomorphisms,

$$\begin{aligned} OSub(X) &\cong \Omega X \\ CSub(X) &\cong \Omega X^{op}. \end{aligned}$$

These allow our intuitions about closed and open subsets to be turned into formulae on opens, i.e. into lattice theory. To give a good example of this we need first to define when a locale is discrete and compact Hausdroff.

Definition 2 *(i) A locale X is* discrete *if it is open and the diagonal $\Delta : X \hookrightarrow X \times X$ is an open sublocale.*

(ii) A locale X is compact Hausdorff *if it is compact and the diagonal $\Delta : X \hookrightarrow X \times X$ is a closed sublocale.*

The full subcategory of **Loc** consisting of the discrete locales is equivalent to the category of discrete topological spaces, i.e. to **Set** ([JT84]). The full subcategory consisting of the compact Hausdorff locales is equivalent to **KHausSp** the category of compact Hausdorff topological spaces ([V91] and [J82]). So we have not generalized or specialized by moving to locales, at least as far as these two classes of spaces are concerned. This is worth re-stating. The category of sets is the same thing as the category of discrete locales and the category of compact Hausdorff spaces is the same thing as the category of compact Hausdorff locales.

Both the category **Set** and the category **KHausSp** are *regular*; that is, they have finite limits and pullback stable image factorizations (A1.3 [J02]). **Set** is trivially regular and **KHausSp** is well known to be regular, for example by appealing to Manes's theorem. In any regular category an associative relational composition can be defined using image factorization. The identity of this relational composition is the diagonal (A3.1.1/2 [J02]). Localically we have formulae for image factorization and hence for relational composition, and these formulae will a take central role in what follows:

Proposition 3 *(i) If $f : X \to Y$ is a locale map between discrete locales then $\exists_f : \Omega X \to \Omega Y$ exists and as an action on open sublocales takes $a \hookrightarrow X$ to the image of $a \hookrightarrow X \xrightarrow{f} Y$.*

(ii) If $f : X \to Y$ is a locale map between compact Hausdorff locales then $\forall_f : \Omega X \to \Omega Y$, as an action on closed sublocales, takes $\neg a \hookrightarrow X$ to the image of $\neg a \hookrightarrow X \xrightarrow{f} Y$.

(ii) is demonstrating that the lattice theoretic map $\forall_f$ is carrying the spatial intuition of image factorization for closed sublocales.

Proof. Check the proposition for **Set** and **KHausSp** and then use the fact that discrete spaces and compact Hausdorff spaces are equivalent to discrete locales and compact Hausdorff locales respectively. ■

3 Relational Composition

In this section formulae on opens are developed that express both discrete and compact Hausdorff relational composition.

Certainly if $R_1 \hookrightarrow Y \times X$ and $R_2 \hookrightarrow X \times W$ are open sublocales for discrete Y, X and W we can define their relational composition,

$$R_1; R_2 = \{(j,k) \mid \exists i \text{ with } (j,i) \in R_1 \text{ and } (i,k) \in R_2\}$$

or, expressed as an open

$$R_1; R_2 = \exists_{\pi_{13}}(1 \otimes \Omega\Delta \otimes 1)(R_1 \otimes R_2)$$

since $\exists_{\pi_{13}} : \Omega Y \otimes_{\mathbf{sup}} \Omega X \otimes_{\mathbf{sup}} \Omega W \to \Omega Y \otimes_{\mathbf{sup}} \Omega W$ is image factorization. Now $\exists_{\pi_{13}}$ exists even if Y and W are not necessarily discrete, since we have noted that $\exists_{\pi_{13}} = Id_{\Omega Y} \otimes \exists_{!X} \otimes Id_{\Omega W}$. Although the spatial intuitions may not apply for general such Y and W, we will still use the term relational composition to define this action on sublocales.

In exactly the same manner we can define relational composition for $\neg R_1 \hookrightarrow Y \times X$ and $\neg R_2 \hookrightarrow X \times W$ closed sublocales of compact Hausdorff Y, X and W;

$$\neg R_1; \neg R_2 = \neg \forall_{\pi_{13}}(1 \odot \Omega\Delta \odot 1)(R_1 \odot R_2).$$

This is the correct formula since $\forall_{\pi_{13}}$ is image factorization in the category of compact Hausdorff locales. Similarly to the discrete case there is no requirement that Y and W be compact Hausdorff.

Remark 4 *(Vickers) It is worth checking that this makes sense spatially. If* $R_1 = b \odot a$ *and* $R_2 = \overline{a} \odot c$*, then* $(y,w) \notin \neg R_1; \neg R_2$ *if and only if*

$$(y,z) \notin \neg R_1 \text{ or } (z,w) \notin \neg R_1$$

for all z, i.e.

$$(y,z) \in b \odot a \text{ or } (z,w) \in \overline{a} \odot c$$

for all z. So the complement of $\neg R_1; \neg R_2$ *is the open* $b \odot c \vee \bigvee_{1 \leq a \vee \overline{a}} 1_{\Omega Y \otimes_{\mathbf{PreFr}} \Omega W}$*, i.e.* $\forall_{\pi_{13}}(1 \odot \Omega\Delta \odot 1)(R_1 \odot R_2)$ *as required.*

We are now in a position to prove a central technical lemma. The suplattice version of this lemma, (i), is basic locale theory and the preframe version, (ii), is a key technical step in [T96]; though note it also appears in [V97]. The expert might recognize (ii) as a corollary to the Hofmann-Mislove theorem carried out in the topos of sheaves over W.

Lemma 5 *Let W be an arbitrary locale then*

(i) For any discrete locale X there is an order isomorphism

$$OSub(X \times W) \cong \mathbf{sup}(\Omega X, \Omega W)$$

and relational composition maps to function composition.

(ii) For any compact Hausdorff locale X

$$CSub(X \times W)^{op} \cong \mathbf{PreFr}(\Omega X, \Omega W)$$

and relational composition maps to function composition.

Given a relation $R \hookrightarrow X \times W$ we will send it to some $\psi_R : \Omega X \to \Omega W$ in the proof to follow. The assertion 'relational composition maps to function composition' is the requirement that $\psi_R \psi_{R'} = \psi_{R';R}$ where $R' \hookrightarrow Y \times X$ is some other relation with Y discrete (or compact Hausdorff for part (ii)).

Proof. (i) Given $R \hookrightarrow X \times W$ send $a \hookrightarrow X$ to $a; R \hookrightarrow W$. As formulae on opens this amount to defining $\psi_R : \Omega X \to \Omega W$ to be

$$a \longmapsto \exists_{\pi_2}(\Omega\pi_1(a) \wedge R),$$

clearly a suplattice homomorphism. Since ψ_R is defined via relational composition it is clear that relational composition maps to function composition by associativity of relational composition.

In the other direction send any suplattice homomorphism $\psi : \Omega X \to \Omega W$ to the open $R_\psi = (Id_{\Omega X} \otimes \psi)(\exists_\Delta(1))$. Now for any $a \in \Omega X$,

$$a = \exists_{\pi_2}(\Omega\pi_1(a) \wedge \exists_\Delta(1))$$

this is because $a; \Delta = a$ as Δ is the identity with respect to relational composition. But for the projection $\pi_2 : X \times X \to X$ in this last equation we have that $\exists_{\pi_2} = \exists_{!X} \otimes Id_{\Omega X}$ and so by applying ψ we obtain

$$\begin{aligned} \psi(a) &= \psi(\exists_{!X} \otimes Id_{\Omega X})[(\Omega\pi_1(a) \wedge \exists_\Delta(1)] \\ &= \exists_{\pi_2}(Id_{\Omega X} \otimes \psi)[\Omega\pi_1(a) \wedge \exists_\Delta(1)] \\ &= \exists_{\pi_2}(\Omega\pi_1(a) \wedge (Id_{\Omega X} \otimes \psi)[\exists_\Delta(1)]) \end{aligned}$$

where the last line follows from recalling that $\Omega\pi_1(a) = a \otimes 1$. Therefore $\psi = \psi_{R_\psi}$.

Finally, since $R = \Delta; R$, it is sufficient to check that $(Id_{\Omega X} \otimes \psi_R)(\overline{R}) = \overline{R}; R$ for any $\overline{R} \hookrightarrow X \times X$. Now for $\pi_{13} : X \times X \times W \to X \times W$ we have

$$\begin{aligned} \exists_{\pi_{13}} &= (Id_{\Omega X} \otimes \exists_{!X} \otimes Id_{\Omega W}) \\ &= Id_{\Omega X} \otimes \exists_{\pi_2} \end{aligned}$$

and so it is sufficient to verify

$$(1 \otimes \Omega\Delta \otimes 1)(\overline{R} \otimes R) = (Id_{\Omega X} \otimes [\Omega\pi_1(\_) \wedge R])(\overline{R}).$$

This can be done by looking are suplattice generators, i.e. checking for the case $\overline{R} = a_1 \otimes a_2$, $R = a \otimes b$, or noting that the map

$$\begin{aligned} & (Id_{\Omega X} \otimes [\Omega\pi_1(\_) \wedge R])(\overline{R}) \\ = \; & (Id_{\Omega X} \otimes \Omega\Delta_{X \times W})(Id_{\Omega X} \otimes \Omega\pi_1 \otimes Id_{\Omega X} \otimes Id_{\Omega W})(\overline{R} \otimes R) \\ = \; & (1 \otimes \Omega\Delta \otimes 1)(\overline{R} \otimes R) \end{aligned}$$

where the third line is clear since $1 \times \Delta \times 1 : X \times X \times W \to X \times X \times X \times W$ factors as

$$X \times X \times W \overset{1 \times \Delta_{X \times W}}{\longrightarrow} X \times (X \times W) \times (X \times W) \overset{1 \times \pi_1 \times 1 \times 1}{\longrightarrow} X \times X \times X \times W.$$

It is clear, from construction, that this bijection preserves order and we have an order isomorphism as required.

(ii) The proof is exactly the preframe parallel. It is included for completeness.

Given $\urcorner R \hookrightarrow X \times W$ send $\urcorner a \hookrightarrow X$ to $\urcorner a ; \urcorner R \hookrightarrow W$. As formulae on opens this amount to defining $\psi_R : \Omega X \to \Omega W$ to be

$$a \longmapsto \forall_{\pi_2}(\Omega\pi_1(a) \vee R),$$

clearly a preframe homomorphism. Since ψ_R is defined via relational composition it is clear that relational composition maps to function composition by associativity of relation composition.

In the other direction send any preframe homomorphism $\psi : \Omega X \to \Omega W$ to the open $R_\psi = (Id_{\Omega X} \odot \psi)(\forall_\Delta(0))$. Now for any $a \in \Omega X$,

$$a = \forall_{\pi_2}(\Omega\pi_1(a) \wedge \forall_\Delta(0))$$

this is because $a; \Delta = a$ as Δ is the identity with respect to relational composition. But for the projection $\pi_2 : X \times X \to X$ in this last equation we have that $\forall_{\pi_2} = \forall_{!X} \odot Id_{\Omega X}$ and so by applying ψ we obtain

$$\begin{aligned} \psi(a) &= \forall_{\pi_2}(Id_{\Omega X} \odot \psi)(\Omega\pi_1(a) \vee \forall_\Delta(1)) \\ &= \forall_{\pi_2}(\Omega\pi_1(a) \vee (Id_{\Omega X} \odot \psi)[\forall_\Delta(1)]) \end{aligned}$$

where the last line follows from recalling that $\Omega\pi_1(a) = a \odot 0$. Therefore $\psi = \psi_{R_\psi}$.

Finally, since $\urcorner R = \Delta; \urcorner R$, it is sufficient to check that $\urcorner(Id_{\Omega X} \odot \psi_R)(\overline{R}) = \urcorner\overline{R}; \urcorner R$ for any $\urcorner\overline{R} \hookrightarrow X \times X$. Now for $\pi_{13} : X \times X \times W \to X \times W$ we have

$$\begin{aligned} \forall_{\pi_{13}} &= (Id_{\Omega X} \odot \forall_{!X} \odot Id_{\Omega W}) \\ &= Id_{\Omega X} \odot \forall_{\pi_2} \end{aligned}$$

and so it is sufficient to verify

$$(1 \odot \Omega\Delta \odot 1)(\overline{R} \odot R) = (Id_{\Omega X} \odot [\Omega\pi_1(\_) \vee R])(\overline{R}).$$

This can be done by looking are preframe generators, i.e. checking the case when $\overline{R} = a_1 \odot a_2$ and $R = a \odot b$, or noting that

$$\begin{aligned} & (Id_{\Omega X} \odot [\Omega\pi_1(\_) \vee R])(\overline{R}) \\ = \ & (Id_{\Omega X} \odot \Omega\Delta_{X \times W})(Id_{\Omega X} \odot \Omega\pi_1 \odot Id_{\Omega X} \odot Id_{\Omega W})(\overline{R} \odot R) \\ = \ & (1 \odot \Omega\Delta \odot 1)(\overline{R} \odot R). \end{aligned}$$

■

In (i) when W is, along with X, also discrete then the result is a well known exercise: For any sets X and W, subsets of $X \times W$ are in bijection with suplattice homomorphisms $PX \to PW$. If further $W = X$ we can use the lemma to give a characterization of the main properties of relations. For example a relation $R \subseteq X \times X$ is

$$\begin{aligned} \text{(i) } \textit{reflexive} \text{ iff } \psi_R &\geq Id \\ \text{(ii) } \textit{transitive} \text{ iff } \psi_R\psi_R &\leq \psi_R \text{ and} \\ \text{(iii) } \textit{anti-symmetric} \text{ iff } \exists_\Delta(1) &\geq (\psi_{R^{op}} \otimes Id)(\exists_\Delta(1)) \wedge (\psi_R \otimes Id)(\exists_\Delta(1)). \end{aligned}$$

If R is reflexive, transitive and anti-symmetric then we of course use the notation $\leq_X$ for R, and use $\uparrow$ (respectively $\downarrow$) for ψ_R (respectively $\psi_{R^{op}}$).

The situation for compact Hausdroff X is the same but the directions of the inequalities are reversed as the bijection of (ii) is order reversing. A closed sublocale $\neg R \hookrightarrow X \times X$ is

$$\begin{aligned} \text{(i) } \textit{reflexive} \text{ iff } \psi_R &\leq Id \\ \text{(ii) } \textit{transitive} \text{ iff } \psi_R\psi_R &\geq \psi_R \text{ and} \\ \text{(iii) } \textit{anti-symmetric} \text{ iff } \forall_\Delta(0) &\leq (\psi_{R^{op}} \odot Id)(\forall_\Delta(0)) \vee (\psi_R \odot Id)(\forall_\Delta(0)). \end{aligned}$$

An *ordered compact Hausdorff locale* is, by definition, a compact Hausdorff locale X together a closed relation $\leq \hookrightarrow X \times X$ which is reflexive symmetric and transitive. The notation $\Uparrow^{op}$ (respectively $\Downarrow^{op}$) is then used for $\psi_\leq$ (respectively $\psi_\geq$). This is by analogy with category theory where if $F : \mathcal{C} \to \mathcal{D}$ is some functor then the notation $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ can be used for the same functor but acting on the dual categories.

As a final example of the applications of this lemma, let us see how it can be used to turn spatial intuitions about locales into true statements about suplattice and preframe homomorphisms. Say K and I are some subsets of a partially ordered set $(X, \leq)$. Then

$$(\exists i \in \uparrow K \cap \downarrow I) \Longrightarrow (\exists k \in K \cap \downarrow I)$$

is certainly a true statement about the elements of X. Now this statement can be expressed in terms of relational composition as it is saying exactly that the relation $\uparrow K \hookrightarrow 1 \times X$ when composed with $\downarrow I \hookrightarrow X \times 1$ is less then or equal to the composition of $K \hookrightarrow 1 \times X$ followed by $\downarrow I \hookrightarrow X \times 1$. But since $\uparrow K = K; \leq$ and $\downarrow I = \leq; I$ this is immediate by the idempotency of $\leq$ with respect to relational composition. The corresponding statement about suplattice homomorphisms reads:

$$\exists_{!X} \Omega\Delta(\uparrow K \otimes \downarrow I) \leq \exists_{!X} \Omega\Delta(K \otimes \downarrow I).$$

Since this is a consequence of an argument involving relational composition we can apply it to compact Hausdorff $(X, \leq)$ to obtain

$$\forall_{!X} \Omega\Delta(\Uparrow^{op} K \odot \Downarrow^{op} I) \geq \forall_{!X} \Omega\Delta(K \odot \Downarrow^{op} I)$$

for any opens K and I in ΩX by using part (ii) of the lemma. Both formulae are used below.

4 Extending Relational Composition to Locales of Upper Sets

This section is mostly technical indicating how the previous lemma specializes when $W = X$ and X is a poset.

If $(X, \leq)$ is a (discrete) poset then the fixed points of $\uparrow : PX \to PX$ are (in order isomorphism with) the upper closed subsets of X. If $(X, \leq)$ is an ordered compact Hausdorff locale then the fixed points of $\Uparrow^{op} : \Omega X \to \Omega X$ are (in order reversing isomorphism with) the upper closed sublocales of X. Both fixed sets are frames, and the most general result that can be called on to show this seems to be:

Lemma 6 *Let X be any locale then*

(i) If $\psi : \Omega X \to \Omega X$ is an idempotent suplattice homomorphism then the fixed set of ψ is a frame.

(ii) If $\psi : \Omega X \to \Omega X$ is an idempotent preframe homomorphism then the fixed set of ψ is a frame.

Proof. (i) Certainly $A \equiv \{a \in \Omega X \mid \psi(a) = a\}$ is a subsuplattice of ΩX. Given $a, b \in A$ we have that $a \wedge_A b = \psi(a \wedge_{\Omega X} b)$ by an easy calculation. For any subsets A_0 of A and any $b \in A$

$$\begin{aligned}(\bigvee_{a \in A_0} a) \wedge_A b &= \psi((\bigvee_{a \in A_0} a) \wedge_{\Omega X} b) \\ &= \psi(\bigvee_{a \in A_0} a \wedge_{\Omega X} b) \\ &= \bigvee_{a \in A_0} \psi(a \wedge_{\Omega X} b) \\ &= \bigvee_{a \in A_0} a \wedge_A b\end{aligned}$$

and so the infinitary distributivity law holds as is required to prove A is a frame.

(ii) Almost identical argument. Following the same notation as in (i), for $a, b \in A$ we have that $a \vee_A b = \psi(a \vee_{\Omega X} b)$. It remains to check the finite distributivity for A and this follows the same pattern as the infinite distributivity calculation just given. ■

Certainly $\uparrow : PX \to PX$ and $\Uparrow^{op} : \Omega X \to \Omega X$ are idempotent given that partial orders are both reflexive and transitive and the sets of their fixed points are then frames by the lemma. We use the (standard) notation $\mathcal{U}X$ for the fixed points of $\uparrow : PX \to PX$ and the notation $\Omega\overline{X}$ for the fixed points of $\Uparrow^{op} : \Omega X \to \Omega X$.

When creating $\mathcal{U}X$ we are splitting an idempotent. So $\uparrow: PX \to PX$ factors as a suplattice surjection follows by a suplattice inclusion, denoted, say

$$q_\uparrow \;:\; PX \twoheadrightarrow \mathcal{U}X$$
$$i_\uparrow \;:\; \mathcal{U}X \hookrightarrow PX$$

where $q_\uparrow i_\uparrow = Id_{\mathcal{U}X}$. Thus $\mathcal{U}X$ is both a split suplattice quotient and a split subsuplattice of PX. This has application key to our considerations. For any locale W, suplattice homomorphisms $\phi : \Omega W \to \mathcal{U}X$ are in bijection with suplattice homomorphisms $\overline{\phi} : \Omega W \to PX$ which enjoy $\uparrow \overline{\phi} = \overline{\phi}$; this is using the fact that $\mathcal{U}X$ is a split inclusion. On the other hand suplattice homomorphisms $\phi : \mathcal{U}X \to \Omega W$ are in bijection with suplattice homomorphisms $\overline{\phi} : PX \to \Omega W$ which enjoy $\overline{\phi} \uparrow = \overline{\phi}$; this is using the fact that $\mathcal{U}X$ is a split surjection. These bijections are found by composition with $q_\uparrow$ or $i_\uparrow$ and so are necessarily order isomorphisms. Taking $\Omega W = \mathcal{U}X$ in both observations we get:

Theorem 7 *(i) For any poset $(X, \leq_X)$, there is an order isomorphism*

$$\mathcal{U}(X^{op} \times X) \cong \mathbf{sup}(\mathcal{U}X, \mathcal{U}X).$$

(ii) For any ordered compact Hausdorff locale, $(X, \leq)$, there is an order isomorphism

$$\Omega(\overline{X^{op} \times X}) \cong \mathbf{PreFr}(\Omega\overline{X}, \Omega\overline{X}).$$

Proof. (i) The preamble establishes an order isomorphism between $\mathbf{sup}(\mathcal{U}X, \mathcal{U}X)$ and relations $R \hookrightarrow X \times X$ which enjoy

$$R = \leq_X; R; \leq_X .$$

But such R are in bijection with the 'upper closed' subsets of $X \times X$ for the ordering $(i, j) \leq (i', j')$ iff $i' \leq_X i$ and $j \leq_X j'$; we are using the notation $X^{op} \times X$ to refer to $X \times X$ with this ordering and so are done.

(ii) Identical argument since the splitting of $\Uparrow^{op}: \Omega X \to \Omega X$ exhibits $\Omega\overline{X}$ both as a split subpreframe and as a split preframe quotient. ■

Notice that under the bijection in (i), morphisms $\psi : \mathcal{U}X \to \mathcal{U}X$ map to $(1 \otimes i_\uparrow \psi q_\uparrow)(\exists_\Delta(1))$ as we are specializing the bijection $OSub(X \times W) \cong \mathbf{sup}(\Omega X, \Omega W)$ given in the previous section. So, for example, the identity on $\mathcal{U}X$ maps to the relation $\leq_X \hookrightarrow X \times X$. Identical comments apply to compact Hausdorff X.

As a final application by taking $W = 1$ in (i) and (ii) of the previous lemma one obtains,

$$\mathcal{U}X^{op} \cong \mathbf{sup}(\mathcal{U}X, \Omega)$$

and

$$\Omega\overline{X^{op}} \cong \mathbf{PreFr}(\Omega\overline{X}, \Omega)$$

the former being well known (for example being an aspect of the self duality of the category of suplattice, [JT84]), the latter being, essentially, Lawson duality

([L79]). Using the former notice that

$$\begin{aligned}\mathbf{sup}(\mathcal{U}X \otimes_{\mathbf{sup}} \mathbf{sup}(\mathcal{U}X,\Omega),\Omega) &\cong \mathbf{sup}(\mathbf{sup}(\mathcal{U}X,\Omega),\mathbf{sup}(\mathcal{U}X,\Omega)) \quad (*)\\ &\cong \mathbf{sup}(\mathcal{U}X^{op},\mathcal{U}X^{op})\\ &\cong \mathcal{U}(X\times X^{op})\end{aligned}$$

and so the evaluation map corresponds to a relation on $X \times X^{op}$. Since the evaluation map is, by definition, the image of the identity $Id_{\mathbf{sup}(\mathcal{U}X,\Omega)}$ under the bijection $(*)$, it is clear that the evaluation map, under these bijections, corresponds to the relation $\geq_X \hookrightarrow X \times X^{op}$. This is key as it gives us a representation of the order relation which is available using only the frame $\mathcal{U}X$. In the same manner there is a representative for the partial order on any compact Hausdorff X, using only the frame $\Omega\overline{X}$. In practice however we will find that the following representative of the evaluation map in terms of relational composition is the most useful:

Lemma 8 *(i) If $(X,\leq)$ is a poset then under the bijection $\mathcal{U}X^{op} \cong \mathbf{sup}(\mathcal{U}X,\Omega)$ the evaluation map is given by*

$$\begin{aligned}\mathcal{U}(X^{op}) \otimes_{\mathbf{sup}} \mathcal{U}(X) &\to \Omega\\ I \otimes J &\longmapsto \exists_{!X}(I \wedge J).\end{aligned}$$

(ii) If $(X,\leq)$ is an ordered compact Hausdorff locale then under the bijection $\Omega\overline{X^{op}} \cong \mathbf{PreFr}(\Omega\overline{X},\Omega)$ the evaluation map is given by

$$\begin{aligned}\Omega\overline{X^{op}} \otimes_{\mathbf{PreFr}} \Omega\overline{X} &\to \Omega\\ I \odot J &\longmapsto \forall_{!X}(I \vee J).\end{aligned}$$

Proof. (i) For $I \in \mathcal{U}X^{op}$, its mate under $\mathcal{U}X^{op} \cong \mathbf{sup}(\mathcal{U}X,\Omega)$ is the map sending $K \hookrightarrow 1 \times X$ to $K;I$, so the evaluation maps sends $I \otimes J$ to $J;I \hookrightarrow 1$. But $\exists_{!X}(I \wedge J)$ is the formulae for this relational composition and so we are done.

(ii) Identical argument. ■

For (i) above note that of course spatially $\exists_{!X}(I \wedge J) = 1$ if and only if $\exists k \in I \cap J$; this will help us argue some motivational spatial reasoning below after the next and final technical lemma:

Lemma 9 *(i) For any poset $(X,\leq)$,*

$$\mathcal{U}(X^{op}) \otimes_{\mathbf{sup}} \mathcal{U}(X) \cong \mathcal{U}(X^{op} \times X).$$

. *(ii) For any ordered compact Hausdorff locale, $(X,\leq)$,*

$$\Omega\overline{X^{op}} \otimes_{\mathbf{PreFr}} \Omega\overline{X} \cong \Omega(\overline{X^{op} \times X}).$$

Proof. (i) The assignment $R \longmapsto \leq;R;\leq$ on R an open sublocale of $X \times X$ defines a suplattice homomorphism for which we will use the notation

$$\psi_{\leq;(\_);\leq} : P(X \times X) \to P(X \times X).$$

The square

$$\begin{array}{ccc} P(X\times X) & \stackrel{\psi_{\leq;(\_);\leq}}{\longrightarrow} & P(X\times X) \\ \cong\downarrow & & \cong\downarrow \\ PX\otimes_{\mathbf{sup}}PX & \stackrel{\downarrow\otimes\uparrow}{\longrightarrow} & PX\otimes_{\mathbf{sup}}PX \end{array}$$

commutes and both the horizontal homomorphisms are idempotent. $\mathcal{U}(X^{op}\times X)$ splits the top homomorphism by definition and $\mathcal{U}(X^{op})\otimes_{\mathbf{sup}}\mathcal{U}(X)$ splits the bottom via $i_\downarrow\otimes i_\uparrow$ and $q_\downarrow\otimes q_\uparrow$. So $\mathcal{U}(X^{op})\otimes_{\mathbf{sup}}\mathcal{U}(X)\cong\mathcal{U}(X^{op}\times X)$ by uniqueness of limits.

(ii) Symmetrically we have a square

$$\begin{array}{ccc} \Omega(X\times X) & \stackrel{\psi_{\leq;(\_);\leq}}{\longrightarrow} & \Omega(X\times X) \\ \cong\downarrow & & \cong\downarrow \\ \Omega X\otimes_{\mathbf{PreFr}}\Omega X & \stackrel{\Downarrow^{op}\otimes\Uparrow^{op}}{\longrightarrow} & \Omega X\otimes_{\mathbf{PreFr}}\Omega X \end{array}$$

for a preframe homomorphism $\psi_{\leq;(\_);\leq}$. ■

5 Retrieving Discrete and Compact Hausdorff Locales from their 'Up-set' Locales

In the introduction it was indicated that the suplattice endomorphism needed to extract PX from $\mathcal{U}(X^{op}\times X)$ was

$$R\longmapsto\leq;(R\cap\Delta);\leq$$

Since $\mathcal{U}(X^{op}\times X)\cong\mathcal{U}(X^{op})\otimes_{\mathbf{sup}}\mathcal{U}(X)$, it should be clear that this endomorphism is equivalent to

$$\begin{aligned}\Psi\quad:\quad&\mathcal{U}(X^{op})\otimes_{\mathbf{sup}}\mathcal{U}(X)\stackrel{i_\downarrow\otimes i_\uparrow}{\hookrightarrow}PX\otimes_{\mathbf{sup}}PX\stackrel{\Omega\Delta}{\to}\\&PX\stackrel{\exists\Delta}{\to}PX\otimes_{\mathbf{sup}}PX\stackrel{q_\downarrow\otimes q_\uparrow}{\longrightarrow}\mathcal{U}(X^{op})\otimes_{\mathbf{sup}}\mathcal{U}(X)\end{aligned}$$

In broad terms we have clarified that this endomorphism can be expressed without reference to X and $\uparrow$, since we have seen above that the evaluation map represents the partial order. The following proposition proves this broad assertion in detail.

Lemma 10 *(i) Given a poset $(X,\leq)$, under the bijections*

$$\begin{aligned}&\mathbf{sup}(\mathcal{U}(X^{op})\otimes_{\mathbf{sup}}\mathcal{U}(X),\mathcal{U}(X^{op})\otimes_{\mathbf{sup}}\mathcal{U}(X))\\\cong\;&\mathbf{sup}(\mathcal{U}(X^{op})\otimes_{\mathbf{sup}}\mathcal{U}(X),\mathbf{sup}(\mathcal{U}X,\mathcal{U}X))\\\cong\;&\mathbf{sup}(\mathcal{U}(X^{op})\otimes_{\mathbf{sup}}\mathcal{U}(X)\otimes_{\mathbf{sup}}\mathcal{U}(X),\mathcal{U}X)\\\cong\;&\mathbf{sup}(\mathcal{U}(X^{op})\otimes_{\mathbf{sup}}\mathcal{U}(X)\otimes_{\mathbf{sup}}\mathcal{U}(X),\mathbf{sup}(\mathcal{U}X^{op},\Omega))\\\cong\;&\mathbf{sup}(\mathcal{U}(X^{op})\otimes_{\mathbf{sup}}\mathcal{U}(X)\otimes_{\mathbf{sup}}\mathcal{U}(X)\otimes_{\mathbf{sup}}\mathcal{U}(X^{op}),\Omega))\end{aligned}$$

the mate of Ψ*, given in the preamble, is*

$$\widetilde{\Psi} : \mathbf{sup}(\mathcal{U}X, \Omega) \otimes_{\mathbf{sup}} \mathcal{U}(X) \otimes_{\mathbf{sup}} \mathcal{U}(X) \otimes_{\mathbf{sup}} \mathbf{sup}(\mathcal{U}X, \Omega) \to \Omega$$
$$I \otimes J \otimes \overline{J} \otimes \overline{I} \longmapsto ev(I \wedge \overline{I} \otimes J \wedge \overline{J}). \tag{1}$$

Proposition 11 *(ii) Similarly given* $(X, \leq)$ *an ordered compact Hausdorff locale the mate of*

$$\Psi \quad : \quad \Omega\overline{X^{op}} \otimes_{\mathbf{PreFr}} \Omega\overline{X} \overset{i_{\Downarrow^{op}} \otimes i_{\Uparrow^{op}}}{\hookrightarrow} \Omega X \otimes_{\mathbf{PreFr}} \Omega X \overset{\Omega\Delta}{\to}$$
$$\Omega X \overset{\forall_\Delta}{\to} \Omega X \otimes_{\mathbf{PreFr}} \Omega X \overset{q_{\Downarrow^{op}} \otimes q_{\Uparrow^{op}}}{\longrightarrow} \Omega\overline{X^{op}} \otimes_{\mathbf{PreFr}} \Omega\overline{X}$$

is

$$\widetilde{\Psi} : \mathbf{PreFr}(\Omega\overline{X}, \Omega) \otimes_{\mathbf{PreFr}} \Omega\overline{X} \otimes_{\mathbf{PreFr}} \Omega\overline{X} \otimes_{\mathbf{PreFr}} \mathbf{PreFr}(\Omega\overline{X}, \Omega) \to \Omega$$
$$I \odot J \odot \overline{J} \odot \overline{I} \longmapsto ev(I \vee \overline{I} \odot J \vee \overline{J}).$$

Dealing with (i) let us first show that $\widetilde{\Psi}$ is the mate of Ψ by appealing to a spatial argument. This will help to motivate the proof to follow. Now $\widetilde{\Psi}$ is corresponds to the relation

$$\mathbf{R} \subseteq X \times X^{op} \times X^{op} \times X$$

where $(i, j, \overline{j}, \overline{i}) \in \mathbf{R}$ if and only if $\exists k \in \uparrow i \cap \uparrow \overline{i} \cap \downarrow j \cap \downarrow \overline{j}$. The suplattice endomorphism, Ψ, on $\mathcal{U}(X^{op} \times X)$ is uniquely determined by a monotone map $\sigma : X \times X^{op} \to \mathcal{U}(X^{op} \times X)$ where

$$\begin{aligned} \sigma(i,j) &= \leq; ([\downarrow i \times \uparrow j] \cap \Delta); \leq \\ &= \bigcup\nolimits_{\exists k \in \downarrow i \cap \uparrow j} \downarrow k \times \uparrow k. \end{aligned}$$

But the subset $\bigcup_{\exists k \in \downarrow i \cap \uparrow j} \downarrow k \times \uparrow k$, as a monotone map from $X^{op} \times X \to \Omega$ is

$$(\overline{i}, \overline{j}) \longmapsto 1 \text{ iff } \ \exists k \in \uparrow i \cap \uparrow \overline{i} \cap \downarrow j \cap \downarrow \overline{j}$$

and so we are done spatially.

Proof. (i) The endomorphism Ψ corresponds to a relation $R_\Psi \hookrightarrow X \times X^{op} \times X^{op} \times X$. To prove that Ψ is the mate of $\widetilde{\Psi}$ it is sufficient to show $\widetilde{\Psi}$ is the mate of R_Ψ under $\mathcal{U}(X \times X^{op} \times X^{op} \times X) \cong \mathbf{sup}(\mathcal{U}(X^{op} \times X \times X \times X^{op}) \to \Omega)$. I.e. that

$$\begin{aligned} & \exists_{!X \times X \times X \times X} (R_\Psi \wedge (I \otimes J \otimes \overline{J} \otimes \overline{I})) \\ = \ & \exists_{!X} \Omega\Delta_4 (I \otimes J \otimes \overline{J} \otimes \overline{I}) \end{aligned}$$

for any $I \otimes J \otimes \overline{J} \otimes \overline{I} \in \mathcal{U}(X^{op}) \otimes_{\mathbf{sup}} \mathcal{U}(X) \otimes_{\mathbf{sup}} \mathcal{U}(X) \otimes_{\mathbf{sup}} \mathcal{U}(X^{op})$ using Lemma 8.

Now lets say that $R_\Psi = (\uparrow \otimes \downarrow \otimes \downarrow \otimes \uparrow) \exists_{\Delta_4}(1)$ where $\Delta_4 : X \hookrightarrow X \times X \times X \times X$. Then $(\downarrow \otimes \uparrow \otimes \downarrow \otimes \uparrow)(I \otimes J \otimes \overline{J} \otimes \overline{I}) = (I \otimes J \otimes \overline{J} \otimes \overline{I})$ and so

$$\exists_{!X \times X \times X \times X} (R_\Psi \wedge (I \otimes J \otimes \overline{J} \otimes \overline{I})) \leq \exists_{!X \times X \times X \times X} (\exists_{\Delta_4}(1) \wedge (I \otimes J \otimes \overline{J} \otimes \overline{I}))$$

by the final comments of Section 3. The opposite inequality is immediate since $(\downarrow \otimes \uparrow \otimes \downarrow \otimes \uparrow) \geq Id$. But for any open sublocale $i_a : a \hookrightarrow X$ we have that $\exists_{i_a}\Omega(i_a) = \exists_{i_a}(1) \wedge a$ and so

$$\begin{aligned} & \exists_{!X\times X\times X\times X}(\exists_{\Delta_4}(1) \wedge (I \otimes J \otimes \overline{J} \otimes \overline{I})) \\ = \; & \exists_{!X\times X\times X\times X}\exists_{\Delta_4}\Omega\Delta_4(I \otimes J \otimes \overline{J} \otimes \overline{I}) \\ = \; & \exists_{!X}\Omega\Delta_4(I \otimes J \otimes \overline{J} \otimes \overline{I}) \end{aligned}$$

where the last line follows since $\exists_{!X\times X\times X\times X}\exists_{\Delta_4}$ is left adjoint to $\Omega\Delta_4\Omega!^{X\times X\times X\times X} = \Omega!^X$.

It remains to show that indeed $R_\Psi = (\uparrow \otimes \downarrow \otimes \downarrow \otimes \uparrow)\exists_{\Delta_4}(1)$. Certainly,

$$(\uparrow \otimes \downarrow \otimes \downarrow \otimes \uparrow)\exists_{\Delta_4}(1) = \leq_{X^{op}\times X}; \exists_{\Delta_4}(1); \leq_{X^{op}\times X}$$

so in order to prove that $R_\Psi = (\uparrow \otimes \downarrow \otimes \downarrow \otimes \uparrow)\exists_{\Delta_4}(1)$ it is sufficient to prove that, via relational composition, they define the same endomorphism on $\mathcal{U}(X^{op}) \otimes_{\mathbf{sup}} \mathcal{U}(X)$; that is we must show that

$$\begin{aligned} I \otimes J; \; & \leq_{X^{op}\times X}; \exists_{\Delta_4}(1); \leq_{X^{op}\times X} \\ & = I \otimes J; R_\Psi \end{aligned}$$

for any $I \otimes J \in \mathcal{U}(X^{op}) \otimes_{\mathbf{sup}} \mathcal{U}(X)$. Since $I \otimes J = I \otimes J; \leq_{X^{op}\times X}$ and $I \otimes J; R_\Psi = \Psi(I \otimes J) = (\downarrow \otimes \uparrow)\exists_\Delta\Omega\Delta(I \otimes J)$ it remains to prove

$$I \otimes J; \exists_{\Delta_4}(1); \leq_{X^{op}\times X} = (\downarrow \otimes \uparrow)\exists_\Delta\Omega\Delta(I \otimes J)$$

and to see this it is in fact sufficient to prove $I \otimes J; \exists_{\Delta_4}(1) = \exists_\Delta\Omega\Delta(I \otimes J)$ because $(\downarrow \otimes \uparrow)(\overline{R}) = \overline{R}; \leq_{X^{op}\times X}$ for any $\overline{R} \hookrightarrow X \times X$. But with $\pi_1, \pi_2 : (X \times X) \times (X \times X) \to (X \times X)$ the two projections we have

$$\begin{aligned} I \otimes J; \exists_{\Delta_4}(1) &= \exists_{\pi_2}(\Omega\pi_1(I \otimes J) \wedge \exists_{\Delta_4}(1)) \\ &= \exists_{\pi_2}\exists_{\Delta_4}\Omega\Delta_4\Omega\pi_1(I \otimes J) \\ &= \exists_{\pi_2}\exists_{\Delta_4}\Omega\Delta(I \otimes J) \\ &= \exists_\Delta\Omega\Delta(I \otimes J) \end{aligned}$$

and so are done.

(ii) Identical argument. For example to prove $\neg(I \odot J); \neg\forall_{\Delta_4}(1) = \neg\forall_\Delta\Omega\Delta(I \odot J)$ for any $I \odot J \in \Omega X \otimes_{\mathbf{PreFr}} \Omega X$ we have

$$\begin{aligned} \neg(I \odot J); \neg\forall_{\Delta_4}(1) &= \neg\forall_{\pi_2}(\Omega\pi_1(I \odot J) \vee \forall_{\Delta_4}(1)) \\ &= \neg\forall_{\pi_2}\forall_{\Delta_4}\Omega\Delta_4\Omega\pi_1(I \odot J) \\ &= \neg\forall_{\pi_2}\forall_{\Delta_4}\Omega\Delta(I \odot J) \\ &= \neg\forall_\Delta\Omega\Delta(I \odot J) \end{aligned}$$

just as above in (i). ■

We are now in a position to make the brief outline contained in the introduction precise.

Theorem 12 *(i) For any partially ordered set $(X, \leq)$, PX is order-isomorphic to the fixed points of the suplattice endomorphism*

$$\Psi : \mathcal{U}(X^{op} \times X) \to \mathcal{U}(X^{op} \times X)$$
$$R \longmapsto \leq; (R \wedge \Delta); \leq$$

(ii) For any compact Hausdorff poset, $(X, \leq)$, ΩX is order-isomorphic to the fixed points of the preframe endomorphism

$$\Psi : \Omega\overline{X^{op} \times X} \to \Omega\overline{X^{op} \times X}$$
$$R \longmapsto \leq; (R \wedge \Delta); \leq$$

The bijection $OSub(X \times X) \cong \mathbf{sup}(\Omega X, \Omega X)$, for discrete X, sends a suplattice $\phi : \Omega X \to \Omega X$ to the open

$$(\phi \otimes 1)\exists_\Delta(1).$$

The open $\exists_\Delta(1)$ is the diagonal on X so note that if $R \hookrightarrow X \times X$ is anti-symmetric (i.e. $R \wedge R^{op} \leq \Delta$ in $Sub(X \times X)$) then

$$\exists_\Delta(1) \geq (\downarrow^R \otimes 1)(\exists_\Delta(1)) \cap (\uparrow^R \otimes 1)(\exists_\Delta(1))$$

with equality if, further, R is reflexive. Using $\Delta_{1212} : X \times X \to (X \times X) \times (X \times X)$ for the diagonal given by $\Delta_{1212}(i,j) = (i,j,i,j)$, we then have

$$\exists_\Delta(1) = \Omega\Delta_{1212}(\downarrow \otimes 1 \otimes \uparrow \otimes 1)(\exists_\Delta(1) \otimes \exists_\Delta(1))$$

whenever $\downarrow$ and $\uparrow$ arise from a partial order on X.

Proof. Consider the suplattice homomorphisms

$$\Gamma \ : \ PX \overset{\exists_\Delta}{\to} PX \otimes_{\mathbf{sup}} PX \overset{q_\downarrow \otimes q_\uparrow}{\longrightarrow} \mathcal{U}(X^{op}) \otimes_{\mathbf{sup}} \mathcal{U}(X)$$
$$\Theta \ : \ \mathcal{U}(X^{op}) \otimes_{\mathbf{sup}} \mathcal{U}(X) \overset{i_\downarrow \otimes i_\uparrow}{\hookrightarrow} PX \otimes_{\mathbf{sup}} PX \overset{\Omega\Delta}{\to} PX.$$

It is sufficient to show that $\Theta\Gamma = Id$ and $\Gamma\Theta$ is the endomorphism given by $R \longmapsto \leq; (R \cap \Delta); \leq$, i.e. Ψ in the notation of the previous proposition. The latter is clear from definition. It remains to check

$$\Omega\Delta(\uparrow \otimes \downarrow)\exists_\Delta = Id. \qquad (*)$$

This is immediate from the anti-symmetry and reflexivity of $\leq$ but, as above, so as to make sure an identical argument using preframe homomorphisms is available we argue the case using only relational composition and suplattice homomorphisms.

The proof now essentially involves ensuring that $(*)$ follows from the anti-symmetry property of $\leq$, expressed as an equation on suplattice homomorphisms as we have done in the preamble.

But the suplattice homomorphism $\Omega\Delta(\uparrow \otimes \downarrow)\exists_\Delta : PX \to PX \otimes_{\mathbf{sup}} PX$ gives rise to a relation

$$(\Omega\Delta \otimes 1)(\uparrow \otimes \downarrow \otimes 1)(\exists_\Delta \otimes 1)(\exists_\Delta(1))$$

on $X \times X \times X$. Since the diagonal $X \hookrightarrow X \times X \times X \times X$ is contained in Δ_{1212} (see preamble), we have that

$$(\exists_\Delta \otimes \exists_\Delta)[\exists_\Delta(1)] \leq \exists_\Delta(1) \otimes \exists_\Delta(1)$$

from which $(1 \otimes \tau \otimes 1)(1 \otimes 1 \otimes \exists_\Delta)(\exists_\Delta \otimes 1)[\exists_\Delta(1)] \leq [(\_) \otimes (\_)][\exists_\Delta(1)]$ where $\tau : PX \otimes_{\mathbf{sup}} PX \to PX \otimes_{\mathbf{sup}} PX$ is transposition. The reason for introducing τ is that $(\Omega\Delta \otimes 1)(\uparrow \otimes \downarrow \otimes 1)$ factors as

$$\Omega\Delta_{1212}(\downarrow \otimes 1 \otimes \uparrow \otimes 1)(1 \otimes \tau \otimes 1)((1 \otimes 1 \otimes \exists_\Delta)$$

and so

$$(\Omega\Delta \otimes 1)(\uparrow \otimes \downarrow \otimes 1)(\exists_\Delta \otimes 1)(\exists_\Delta(1)) \leq \exists_\Delta(1)$$

by the description of anti-symmetry given in the preamble. The the opposite inequality is clear since $\leq$ is reflexive and so $\uparrow$ and $\downarrow$ are both inflationary and $\Omega\Delta\exists_\Delta = Id$. Since the assignment $\phi \longmapsto (\phi \otimes 1)\exists_\Delta(1)$ is a bijection for any suplattice homomorphism ϕ, we are done.

(ii) Identical argument. ■

Clearly, given that $\Psi = \Gamma\Theta$ and $\Theta\Gamma = Id$, Ψ is idempotent. Therefore given any frame of the form $\mathcal{U}X$ (resp. $\Omega\overline{X}$) it follows that PX (resp. ΩX) can be found by splitting the idempotent Ψ, i.e. appealing to Lemma 5(i) (resp. (ii)). Moreover Ψ itself can be found using only $\mathcal{U}(X)$ by taking the mate $\widetilde{\Psi}$ (Lemma 10) definable using only the evaluation map. In short $P(X)$ can be recovered using only the data $\mathcal{U}(X)$ and we have the main result:

Theorem 13 *(i) There is a bijection between posets and locales whose frames of opens is of the form $\mathcal{U}(X)$ for some poset X.*

(ii) There is a bijection between ordered compact Hausdorff locales and locales whose frames of opens are of the form $\Omega\overline{X}$ for some ordered compact Hausdroff locale $(X, \leq)$.

Clearly we must now give some topological clarity as to what these, so far informally named, "up-set" locales are. This is the subject of the next section.

6 Topological Interpretation

In this section we now give topological interpretations to the objects under consideration. Whilst both $\mathcal{U}X$ (resp. $\Omega\overline{X}$) seem natural enough objects of study from their definitions it is important to realize that the class of all objects of the form $\mathcal{U}X$ is well known, as is the class of those of the form $\Omega\overline{X}$; they correspond, respectively, to the algebraic dcpos and the sober stably locally compact topological spaces. We now define these terms and prove the correspondence. Aside from the final theorem, all the results in this section are known.

A *directed complete partial order* (dcpo) is a poset with joins for all directed subsets. For example for any poset X, the poset $idl(X)$ of ideals of X (that is of directed and lower closed subsets of X) is a dcpo. In any dcpo Y we may define the way below relation $\ll \subseteq Y \times Y$ by $y_1 \ll y_2$ if $\forall$ directed subsets I of Y,

$$y_2 \leq \bigvee^{\uparrow} I \Rightarrow \exists i \in I \text{ with } y_1 \leq i$$

The notation $I \subseteq^{\uparrow} Y$ is used to indicate when a subset is directed. A dcpo is *algebraic* if it is of the form $idl(P)$ for some poset P; there are various other characterizations of algebraic dcpos. For example, recalling that $y \in Y$ is compact iff $y \ll y$, a dcpo Y is algebraic if and only if every element is the direct join of compact elements less than it. You may verify that $idl(P)$ is the free dcpo on the poset P, where dcpo homomorphisms are directed join preserving maps. Recall that a subset of a dcpo, $U \subseteq Y$ say, is *Scott open* if it is upper closed and $\bigvee^{\uparrow} I \in U \Rightarrow \exists i \in I \cap U$. Notice that for any dcpo Y there is an order isomorphism between the Scott opens and $\mathbf{dcpo}(Y, \Omega)$ with the obvious pointwise ordering. Therefore for any poset X,

$$\begin{aligned} \mathbf{dcpo}(idl(X), \Omega) &\cong \mathbf{Pos}(X, \Omega) \\ &\cong \mathcal{U}X \end{aligned}$$

and we see that $\mathcal{U}X$ is an isomorphic copy of the Scott topology on $idl(X)$. In the other direction it is well known and easy to check that $\mathcal{U}X$ is the free suplattice on the poset X^{op}. In fact there is an order isomorphism $\mathbf{sup}(\mathcal{U}X, \Omega) \cong \mathbf{Pos}(X^{op}, \Omega)$ and further this specializes to

$$\begin{aligned} \mathbf{Fr}(\mathcal{U}X, \Omega) \cong \{\chi : X^{op} \longrightarrow \Omega \mid \chi \text{ monotone}, \\ \chi(x) = 1, \chi(y) = 1 \implies \exists z \geq x, y,\ \chi(z) = 1\} \end{aligned}$$

I.e. $\mathbf{Fr}(\mathcal{U}X, \Omega) \cong idl(X)$. Thus an algebraic dcpo is completely determined by $\mathcal{U}X$ and the points of the locale corresponding to $\mathcal{U}X$ are in order isomorphism with the ideals of X. For this reason,

Definition 14 *The* ideal completion locale *of a poset X is denoted $Idl(X)$ and is defined by $\Omega Idl(X) \equiv \mathcal{U}X$.*

The main result of the previous section therefore establishes a bijection between ideal completion locales and discrete localic posets. For more information on algebraic dcpos, and their significance to domain theory, consult [V89].

Turning our attention to $\Omega\overline{X}$, i.e. to the compact Hausdorff side, recall that a topological space (X, τ) gives rise to a locale since τ is a frame. In the other direction for any locale X there is a natural topology on $ptX \equiv \mathbf{Loc}(1, X)$ given by sets of the form $\{p \mid \Omega p(a) = 1\}$ over $a \in \Omega X$. A topological space is sober if these two operations are mutually inverse. Sobriety is weaker than Hausdorffness, and in this weaker context we need to adapt the definition of local compactness. Traditionally (e.g. 2.3 of [R66]) a space, (X, τ), is locally compact if for every $x \in X$ is contained in an open V_x whose closure, $\overline{V_x}$, is compact. For Hausdorff X use 2.7 of [R66] to note that this is equivalent to requiring that for any $x \in U$, for any $U \in \tau$ there exists an open V_x with

$$x \in V_x \subseteq \overline{V_x} \subseteq U.$$

such that $\overline{V_x}$ is compact. We shall take this latter condition to be our definition of *local compactness* for an arbitrary topological space. A locale, X, is *locally compact* if

$$a = \bigvee\nolimits^{\uparrow} \{b \mid b \ll a\}$$

for every $a \in \Omega X$. In other words, ΩX is a continuous poset, VII [J82].

Theorem 15 *The category of sober locally compact spaces is equivalent to the category of locally compact locales.*

Proof. We have outlined the functors involved in the definition just given of sober. The situation $V_x \subseteq \overline{V_x} \subseteq U$ given in the definition of locally compact for a space, clearly implies $V_x \ll U$ in τ by compactness of $\overline{V_x}$ and so τ is a continuous poset showing that the corresponding locale is locally compact. The other direction is well known and requires a choice principle. Consult VII 4 [J82]. ■

To define the concept of a sober stably locally compact space the definition of saturation is needed. Every topological space has a *specialization preorder* on it given by $x \sqsubseteq y$ if $x \in U \Longrightarrow y \in U \ \forall U \in \tau$. The *saturation* of any subset X_0 of a topological space is its upper closure with respect to the specialization order; note that this can equivalently be calculated as $\cap\{U \in \tau \mid X_0 \subseteq U\}$. It follows that the saturation of any open subset is itself and that the saturation of a compact subset is compact since any X_0 is contained in an open U if and only if its saturation is. Note that, given the last theorem,

$$\begin{array}{rcl} V & \ll & U \Longleftrightarrow V \subseteq K \subseteq U \text{ some} \\ & & \text{compact and saturated } K \end{array}$$

for any V and U opens of a sober locally compact space.

A sober locally compact topological space is said to be *stably* locally compact (or just *stably compact* by some authors) if, in addition,

(i) it is compact
(ii) $K_1 \cap K_2$ is compact
whenever the K_is are
compact and saturated.

These spaces have been extensively studied, [H85], [H84], [S92], [KB87] [JS96] and [JKM01].

A locally compact locale X is said to be *stably* locally compact (or *stably compact* by some authors) provided

$$\begin{aligned} (i)\ a &\ll 1 \text{ for all } a \in \Omega X \\ (ii)\ a &\ll b_1,\ a \ll b_2 \\ &\Longrightarrow a \ll b_1 \wedge b_2 \end{aligned}$$

where the second condition is over all triples $a, b_1, b_2 \in \Omega X$. The corresponding frames are known as the *stably continuous* frames. These locales/frames have been extensively studied, see VII 4.6 of [J82] for a textbook account. For example they are the algebras of the prime Wallman compactification functor [W84]. Note that (i) in this definition is equivalent to the requirement $1 \ll 1$ and so, under the bijection of the last theorem, condition (i) for stably locally compact spaces is equivalent to condition (i) for stably locally compact locales. Further,

Theorem 16 *The category of sober stably locally compact spaces is equivalent to the category of stably locally compact locales.*

Proof. It is sufficient to specialize the previous theorem. If X is a sober stably locally compact space then certainly the $\ll$ relation on its opens satisfies part (ii) in the localic definition of stably locally compact. This is immediate from our characterization of $\ll$ just given. For the other direction consult, for example, [H84]. ■

Examples of stably locally compact locales include coherent locales (i.e. locales whose frames of opens are of the form $idl(L)$ for a distributive lattice L) and compact Hausdorff locales. To prove the former note that $I \ll J$ for ideals I, J of L if and only if $I \subseteq \downarrow l \subseteq J$ for some $l \in L$. To prove that compact Hausdorff locales are stably locally compact recall that a locale X is compact Hausdorff if and only if it is compact and *regular*, that is

$$a = \bigvee{}^{\uparrow}\{b \mid b \lhd a\}$$

for every $a \in \Omega X$ where $b \lhd a$ if $\exists c$ with $b \wedge c = 0$ and $a \vee c = 1$ (b is *well inside* a). This correspondence between compact Hausdorff and compact regular is Vermeulen's result, [V91], though note Theorem 3.4.2 of [T96] for a proof using the preframe techniques developed above. For compact Hausdorff X it is an exercise, using regularity, to show that $a \lhd b \Longleftrightarrow a \ll b$. The binary stability property needed of $\ll$ in the definition of stably locally compact is immediate for $\lhd$ and so any compact Hausdorff locale is stably locally compact.

Stably locally compact locales can be characterized as the retracts of coherent locales, or the injective objects in **Loc** with respect to flat inclusions (where an inclusion, i, is *flat* provided $\forall_i$ preserves finite joins). They are relevant to our deliberations since

Proposition 17 *A locale X is stably locally compact if and only if $\Omega X \cong \Omega \overline{Y}$ for some compact Hausdorff ordered locale $(Y, \leq)$.*

This seems to have been first observed spatially (i.e. for topological spaces) in [G80], p.313-4. There we see the correspondence between stably locally compact (sober) spaces and ordered compact Hausdorff posets. The creation of Y from X follows the patch construction. The localic/frame theoretic version of patch is, effectively, initially in [BB88], though they only create a biframe (a short step away from an ordered compact Hausdorff locale). A direct construction of the ordered locale Y from X is in Section 7.6.1 of [T96] using a different, but very similar, construction to the one offered here. Escardó [E01] has a more explicit description of localic patch, and we comment on his construction below.

The proof of this proposition uses the techniques of the previous section so it is natural to pause and collection some facts about $\mathbf{PreFr}(\Omega X, \Omega)$ for any stably locally compact locale X:

Lemma 18 *For a stably locally compact locale X,*

(i) $\Lambda\Omega X \cong \mathbf{PreFr}(\Omega X, \Omega)$ where $\Lambda\Omega X$ is the poset of Scott open filters on ΩX,

(ii) $\Lambda\Omega X$ is the frame of opens of a compact locale with $F \ll H$ iff there exists $a \in H$ such that

$$F \subseteq \uparrow a \subseteq H,$$

(iii) $Q \equiv \{\psi \in \mathbf{PreFr}(\Lambda\Omega X, \Lambda\Omega X) \mid Id \leq \psi$ and $\psi^2 = \psi\}$ is a subpreframe of $\mathbf{PreFr}(\Lambda\Omega X, \Lambda\Omega X)$; and

(iv) Q is compact, i.e. $1 \ll 1$.

Proof. (i) This is by definition of Scott open filter. Note that if $G \subseteq \Omega X$ is a Scott open filter then since $a = \bigvee^{\uparrow}\{b \mid b \ll a\}$ for any $a \in \Omega X$, we have that for any $a \in G$ there exists $b \in G$ with $b \ll a$. Conversely any filter with this property is Scott open.

(ii) $\Lambda\Omega X$ is certainly a preframe; directed join is given by union and finite meet is given by intersection. The least Scott open filter is $\{1\}$ (and this is Scott open by compactness assumption on X). Given any two Scott open filters F and G their join is given by

$$F \vee G = \uparrow \{a \wedge b \mid a \in F \text{ and } b \in G\}$$

and it is then routine to verify the finite distributivity law required for $\Lambda\Omega X$ to be a frame. Note we could also set

$$F \vee G = \mathop{\Uparrow}\{a \wedge b \mid a \in F \text{ and } b \in G\} \; (**)$$

using the stable local compactness of X. Compactness of $\Lambda\Omega X$ is immediate since any Scott open filter is top if and only if it contains $0_{\Omega X}$.

(iii) Directed joins and finite meets are calculated pointwise in $\mathbf{PreFr}(\Lambda\Omega X, \Lambda\Omega X)$ and so one needs to check that the conditions $Id_{\Lambda\Omega x} \leq \psi$ and $\psi^2 = \psi$ are closed under these operations. This is immediate for the condition $Id \leq \psi$. For the idempotency condition, certainly the top map ($\psi(G) = \Omega X$ for all $G \in \Lambda\Omega X$)

is idempotent and for any $G \in \Lambda\Omega X$ and any $\psi_1, \psi_2 \in Q$ we have

$$\begin{aligned}
& (\psi_1 \wedge \psi_2)(\psi_1 \wedge \psi_2)(G) \\
= \ & (\psi_1 \wedge \psi_2)[\psi_1(G) \wedge \psi_2(G)] \\
= \ & \psi_1[\psi_1(G) \wedge \psi_2(G)] \wedge \psi_2[\psi_1(G) \wedge \psi_2(G)] \\
= \ & \psi_1\psi_1(G) \wedge \psi_1\psi_2(G) \wedge \psi_1\psi_2(G) \wedge \psi_2\psi_2(G) \\
\leq \ & \psi_1\psi_1(G) \wedge \psi_2\psi_2(G) \\
= \ & \psi_1(G) \wedge \psi_2(G) = (\psi_1 \wedge \psi_2)(G)
\end{aligned}$$

with the reverse inequality since $Id \leq \psi_1 \wedge \psi_2$. For a directed collection of $\psi_i \in Q$ we have, similarly for any G,

$$\begin{aligned}
& (\bigvee\nolimits_i^{\uparrow} \psi_i)(\bigvee\nolimits_i^{\uparrow} \psi_i)(G) \\
= \ & (\bigvee\nolimits_j^{\uparrow} \psi_j) \bigvee\nolimits_i^{\uparrow} \psi_i(G) \\
= \ & \bigvee\nolimits_j^{\uparrow} \psi_j(\bigvee\nolimits_i^{\uparrow} \psi_i(G)) \\
= \ & \bigvee\nolimits_j^{\uparrow} \bigvee\nolimits_i^{\uparrow} \psi_j\psi_i(G) \\
\leq \ & \bigvee\nolimits_k^{\uparrow} \psi_k\psi_k(G) = \bigvee\nolimits_k^{\uparrow} \psi_k(G) \\
= \ & (\bigvee\nolimits_i^{\uparrow} \psi_i)(G)
\end{aligned}$$

where the penultimate line exploits the fact that for any pair ψ_j ψ_i there is an index k such that ψ_j ,$\psi_i \leq \psi_k$, i.e. exploits the directedness of the collection ψ_i.

(iv) For compactness notice that the map

$$\begin{aligned}
\varphi : \Lambda\Omega X & \rightarrow & Q \\
G & \longmapsto & (H \longmapsto H \vee G)
\end{aligned}$$

preserves the top element (1) and has a right adjoint which is a preframe homomorphism. The right adjoint sends a preframe homomorphism $\psi : \Lambda\Omega X \rightarrow \Lambda\Omega X$ to $\psi(0)$. This is sufficient to show that φ preserves $\ll$ and hence compactness of Q follows from compactness of $\Lambda\Omega X$ as compactness is equivalent to the assertion $1 \ll 1$. ∎

Part (ii) of this lemma, and more, is in [BB88]. Escardó and Karazeris also note (iii) in [E98] and [K97] and (iv) is in [E01], a paper that has helped guide the proof to follow. We now prove the proposition.

Proof. The proof is split into a number of parts that have italicized headings.

$\Omega\overline{Y}$ *is always a stably continuous frame.*

Proving that $\Omega\overline{Y}$ is the frame of opens of a stably locally compact locale is routine. Note that for any $a \in \Omega\overline{Y} \subseteq \Omega Y$,

$$b \ll_{\Omega Y} a \Longrightarrow \Uparrow^{op} b \ll_{\Omega\overline{Y}} a$$

and so continuity of $\Omega\overline{Y}$ follows from continuity of ΩY. Use the fact that

$$b_i = \bigvee\nolimits^{\uparrow} \{\Uparrow^{op} c \mid c \lhd_{\Omega Y} b_i\} \qquad (+)$$

for $b_1, b_2 \in \Omega\overline{Y}$, to show the binary stability property for $\ll_{\Omega\overline{Y}}$. (+) follows from the regularity of ΩY.

Defining ΩY from stably locally compact X.

In the other direction we need to construct compact Hausdorff $(Y, \leq)$ from stably locally compact X. The proof involves applying the construction of the previous section to an arbitrary stably locally compact locale X. In the previous section the relevant preframe homomorphisms needed to extract Y was seen to be

$$\widetilde{\Psi} : \mathbf{PreFr}(\Omega X, \Omega) \otimes_{\mathbf{PreFr}} \Omega X \otimes_{\mathbf{PreFr}} \Omega X \otimes_{\mathbf{PreFr}} \mathbf{PreFr}(\Omega X, \Omega) \to \Omega$$
$$I \odot J \odot \overline{J} \odot \overline{I} \longmapsto ev(I \vee \overline{I} \odot J \vee \overline{J})$$

and passing through the various equivalences we have that this corresponds to

$$\Theta : \Lambda\Omega X \otimes_{\mathbf{PreFr}} \Omega X \to \mathbf{PreFr}(\Lambda\Omega X, \Lambda\Omega X)$$
$$F \odot a \longmapsto (G \longmapsto \{b \mid a \vee b \in F \vee G\}).$$

Given the previous section we know that the frame we are looking for is the image of Θ.

Checking that the image of Θ is compact.

Firstly observe that this image is contained within Q as defined in the previous lemma. To see this, fix any $a \in \Omega X$ and $F \in \Lambda\Omega X$. Then for any $b \in G$ and for any Scott open filter G, $a \vee b \geq 1 \wedge b$, where $1 \in F$ and $b \in G$, i.e. $a \vee b \in F$. It follows that $G \subseteq \{b \mid a \vee b \in F \vee G\}$ and so $\Theta(F \odot a)$ is inflationary. To prove that $\Theta(F \odot a)$ is idempotent it needs to be checked, for any Scott open filter G, that

$$\{b \mid a \vee b \in F \vee [\Theta(F \odot a)](G)\}$$
$$\subseteq \{b \mid a \vee b \in F \vee G\}.$$

Say b is in the left hand side, then there exists $a_1 \in F$ and $\widetilde{b} \in [\Theta(F \odot a)](G)$ such that $a \vee b \geq a_1 \wedge \widetilde{b}$. Since $\widetilde{b} \in [\Theta(F \odot a)](G)$ then $a \vee \widetilde{b} \geq a_2 \wedge b_2$ for some $a_2 \in F$ and $b_2 \in G$. Then

$$\begin{aligned} a \vee b &\geq (a_1 \wedge \widetilde{b}) \vee a \\ &= (a_1 \vee a) \wedge (\widetilde{b} \vee a) \\ &\geq (a_1 \vee a) \wedge a_2 \wedge b_2. \end{aligned}$$

But $(a_1 \vee a) \wedge a_2 \in F$ since F is a filter and so $a \vee b \in F \vee G$ and $b \in \{b \mid a \vee b \in F \vee G\}$ as required.

Checking that the image of Θ is a frame.

Next we show that Θ takes finite joins to finite joins so that its image is a frame. To achieve this it is sufficient to show that

$$\begin{aligned} (a)\ \Theta(0 \odot 0) &= 0_Q, \\ (b)\ \Theta(F_1 \vee F_2 \odot 0) &= \Theta(F_1 \odot 0) \vee_Q \Theta(F_2 \odot 0), \\ (c)\ \Theta(0 \odot a_1 \vee a_2) &= \Theta(0 \odot a_1) \vee_Q \Theta(0 \odot a_2) \text{ and} \\ (d)\ \Theta(F \odot a) &= \Theta(F \odot 0) \vee_Q \Theta(0 \odot a). \end{aligned}$$

(a) is clear since 0_Q is the identity map and $\{b \mid a \vee b \in F \vee G\}$ reduces to G when $a = 0$ and $F = 0$. For (b) say $\psi \in \mathbf{PreFr}(\Lambda\Omega X, \Lambda\Omega X)$ has $\Theta(F_i \odot 0) \leq \psi$ for $i = 1, 2$. Then for any G, $F_i \vee G \leq \psi(G)$, $i = 1, 2$, and so certainly $F_1 \vee F_2 \vee G \leq \psi(G)$, i.e. $\Theta(F_1 \vee F_2 \odot 0)(G) \leq \psi(G)$. The reverse is immediate and so (b) is established. For (c) we finally get to exploit the idempotency conditions placed on Q. Similarly say $\psi \in \mathbf{PreFr}(\Lambda\Omega X, \Lambda\Omega X)$ has $\Theta(0 \odot a_i) \leq \psi$ for $i = 1, 2$. It needs to be shown that $\Theta(0 \odot a_1 \vee a_2)(G) \leq \psi(G)$ for any G. Say $b \in LHS$, then $a_1 \vee a_2 \vee b \in G$, and so $a_2 \vee b \in \Theta(0 \odot a_1)(G)$ and so $a_2 \vee b \in \psi(G)$. Since $\psi(G)$ is a Scott open filter there must exist $c \ll a_2 \vee b$ with $c \in G$. Therefore the Scott open filter $\{d \mid c \ll d\}$ is contained in $\psi(G)$. But then $\psi(\{d \mid c \ll d\}) \leq \psi(G)$ by idempotency of ψ and so since

$$\begin{aligned} \Theta(0 \odot a_2)(\{d &\mid c \ll d\}) \\ &\leq \psi(\{d \mid c \ll d\}) \end{aligned}$$

and certainly $b \in \{b' \mid a_2 \vee b' \in \{d \mid c \ll d\}\}$by choice of c, we have that $b \in \psi(G)$ as required. Checking (d) is similar; say $\Theta(F \odot 0)$ and $\Theta(0 \odot a) \leq \psi$. Then by the former $F \vee G \leq \psi(G)$ and so by idempotency of ψ, $\psi(F \vee G) \leq \psi(G)$, but $\{b \mid a \vee b \in F \vee G\} \leq \psi(F \vee G)$ since $\Theta(0 \odot a) \leq \psi$ and so $\Theta(F \odot a) \leq \psi$.

Therefore Θ takes finite joins to finite joins and its image is a frame. Indeed, as directed joins are calculated in Q, it is the frame of opens of a compact locale by the lemma.

Checking that the image of Θ is a regular.

Fix $Im\Theta$ as notation for the image of Θ, i.e. for $\{\Theta(N) \mid N \in \Lambda\Omega X \otimes_{\mathbf{PreFr}} \Omega X\}$. For regularity of this frame it is sufficient to show that for any $a \ll b$,

$$\Theta(0 \odot a) \lhd_{Im\Theta} \Theta(0 \odot b)$$

and for any $F \ll H$ (in $\Lambda\Omega X$),

$$\Theta(F \odot 0) \lhd_{Im\Theta} \Theta(H \odot 0).$$

For the former, take $F = \{d \mid a \ll d\}$ a Scott open filter. To see that $\Theta(0 \odot a) \wedge \Theta(F \odot 0) = 0$ say $b \in [\Theta(0 \odot a) \wedge \Theta(F \odot 0)](G) = \{b \mid a \vee b \in G\} \cap \{b \mid b \in F \vee G\}$. Then $b \geq a_1 \wedge b_1$ with $a_1 \gg a$ and $b_1 \in G$, and $a \vee b \in G$. Then $(a \vee b) \wedge b_1 \in G$ as G is a filter. But $b \geq (a \vee b) \wedge b_1$ by distributivity and so $b \in G$. Hence $\Theta(0 \odot a) \wedge \Theta(F \odot 0)$ is the identity map which we have already established is bottom in $Im\Theta$. Next

$$\begin{aligned} [\Theta(0 \odot b) \vee \Theta(F \odot 0)](G) &= \Theta(F \odot b)(G) \\ &= \{\widetilde{b} \mid b \vee \widetilde{b} \in F \vee G\} \end{aligned}$$

But $b \vee 0 \in F \vee G$ as $a \ll b$ and so $0 \in \Theta(F \odot b)(G)$ implying that it is the top Scott open filter. Hence $\Theta(0 \odot b) \vee \Theta(F \odot 0) = 1$ and the claim $\Theta(0 \odot a) \lhd_{Im\Theta} \Theta(0 \odot b)$ is established.

For the latter claim, recall from the lemma that $F \ll H$ implies $F \subseteq \uparrow a \subseteq H$ for some $a \in H$. We have a similar analysis: $\Theta(0 \odot a) \wedge \Theta(F \odot 0) = 0$ since $[\Theta(0 \odot a) \wedge \Theta(F \odot 0)](G) = \{b \mid a \vee b \in G\} \cap \{b \mid b \in F \vee G\} = G$. And

$$\begin{aligned}[\Theta(0 \odot a) \vee \Theta(H \odot 0)](G) &= \Theta(H \odot a)(G) \\ &= \{b \mid a \vee b \in H \vee G\}\end{aligned}$$

But $0 \in \{b \mid a \vee b \in H \vee G\}$ since $a \in H$ and so $\Theta(0 \odot a) \vee \Theta(H \odot 0)$ is top. This completes the proof that $Im\Theta$ is regular. Observe that, along the way, we have shown that for any $b \in \Omega X$

$$\Theta(0 \odot b) = \bigvee^{\uparrow} \{\Theta(0 \odot a) \mid \Theta(0 \odot a) \lhd_1 \Theta(0 \odot b)\}$$

where $M \lhd_1 N$ if there exists $G \in \Lambda\Omega X$ with $M \wedge \Theta(G \odot 0) = 0$ and $N \vee \Theta(G \odot 0) = 1$, and further that for any $H \in \Lambda\Omega X$

$$\Theta(H \odot 0) = \bigvee^{\uparrow} \{\Theta(F \odot 0) \mid \Theta(F \odot 0) \lhd_2 \Theta(H \odot 0)\}$$

where $M \lhd_2 N$ if there exists $b \in \Omega X$ with $M \wedge \Theta(0 \odot b) = 0$ and $N \vee \Theta(0 \odot b) = 1$. So $\lhd_1$ and $\lhd_2$ are two refinements of $\lhd_{Im\Theta}$.

Defining a partial order on Y and showing $\Omega X \cong \Omega \overline{Y}$.

Since $Im\Theta$ is compact regular we can, by Vermeulen's result, define a compact Hausdorff locale Y by $\Omega Y \equiv Im\Theta$. To turn Y into a localic poset a preframe endomorphism on ΩY needs to be defined which enjoys the conditions (i)-(iii) given at the end of the section 3. Let us define $\phi_1 : \Omega Y \to \Omega Y$ by

$$\phi_1(N) = \bigvee^{\uparrow} \{\Theta(0 \odot a) \mid \Theta(0 \odot a) \lhd_1 N\}$$

and $\phi_2 : \Omega Y \to \Omega Y$ by

$$\phi_2(N) = \bigvee^{\uparrow} \{\Theta(F \odot 0) \mid \Theta(F \odot 0) \lhd_2 N\}$$

It is routine to verify that these are idempotent and deflationary preframe homomorphisms on ΩY and that ΩX (respectively $\Lambda\Omega X$) is order isomorphic to the fixed points of ϕ_1 (respectively ϕ_2) given the observations just made about $\lhd_1$ and $\lhd_2$. So to complete the proof it is sufficient to prove anti-symmetry for, say, ϕ_1. Firstly we check that ϕ_2 is the 'mate' of ϕ_1, the sense that if ϕ_1 corresponds to the relation R then ϕ_2 is corresponds to the relation R^{op}. Given the bijection of Lemma 5 this amounts to checking

$$(Id \odot \phi_1)(\forall_\Delta(0)) = (\phi_2 \odot Id)(\forall_\Delta(0))$$

We show

$$(Id \odot \phi_1)(\forall_\Delta(0)) \leq (\phi_2 \odot Id)(\forall_\Delta(0))$$

by verifying that

$$^\neg N;^\neg(Id\odot\phi_1)(\forall_\Delta(0)) \geq {}^\neg N;^\neg(\phi_2\odot Id)(\forall_\Delta(0)) \quad \text{(a)}$$

for every $N\in\Omega Y$. (For the full result apply a symmetric argument to show that $(Id\odot\phi_2)(\forall_\Delta(0))\leq(\phi_1\odot Id)(\forall_\Delta(0))$ and then apply the twist isomorphism; we omit these details.) The inequality (a) is equivalent to asserting $\phi_1(N)\leq\forall_{\pi_2}((\phi_2\odot Id)(\forall_\Delta(0))\vee N\odot 0)$. By the definition of ϕ_1 and the fact that $\forall_{\pi_2}$ is right adjoint we are left checking

$$0\odot\Theta(0\odot a)\leq(\phi_2\odot Id)(\forall_\Delta(0))\vee N\odot 0$$

for any $\Theta(0\odot a)\lhd_1 N$. Then by definition of $\lhd_1$ there exists a Scott open filter G such that $\Theta(0\odot a)\wedge\Theta(G\odot 0)=0$ and $N\vee\Theta(G\odot 0)=1$. But $\Theta(0\odot a)\wedge\Theta(G\odot 0)=0$ implies that $[0\odot\Theta(0\odot a)]\wedge[\Theta(G\odot 0)\odot 0]\leq\forall_\Delta(0)$ as $\forall_\Delta$ is right adjoint. Further

$$\begin{array}{rl} & (\phi_2\odot Id)\{[0\odot\Theta(0\odot a)]\wedge[\Theta(G\odot 0)\odot 0]\} \\ \leq & [0\odot\Theta(0\odot a)]\wedge[\Theta(G\odot 0)\odot 0] \end{array}$$

since ϕ_2 fixes $\Theta(G\odot 0)$ and is deflationary. So in fact $[0\odot\Theta(0\odot a)]\wedge[\Theta(G\odot 0)\odot 0]\leq(\phi_2\odot Id)\forall_\Delta(0)$ and so since $1=1\odot 0=(N\vee\Theta(G\odot 0))\odot 0=[N\odot 0]\vee[\Theta(G\odot 0)\odot 0]$

$$\begin{array}{rcl} 0\odot\Theta(0\odot a) & = & [0\odot\Theta(0\odot a)]\wedge 1 \\ & = & [0\odot\Theta(0\odot a)]\wedge[N\odot 0\vee\Theta(G\odot 0)\odot 0] \\ & \leq & (\phi_2\odot Id)(\forall_\Delta(0))\vee N\odot 0 \end{array}$$

as required and we conclude that ϕ_1 is the mate of ϕ_2; they both come from the same relation on R.

To prove anti-symmetry of a closed relation R we may alternatively check that $^\neg N;\Delta\geq{}^\neg N;(R^{op}\wedge R)$ for every $N\in\Omega Y$, since relations are in order isomorphism with preframe homomorphisms. As a formulae on opens this asks that $N\leq\forall_{\pi_2}(a_{R^{op}}\vee a_R\vee N\odot 0)$. Now N is a directed join of finite meets of opens of the form $\Theta(F\odot a)$ by definition. Since $\Theta(F\odot a)=\Theta(F\odot 0)\vee\Theta(0\odot a)$ and since finite meets distribute over finite joins it follows that N is the join of opens of the form $\Theta(F\odot 0)\wedge\Theta(0\odot a)$. Since $\Theta(F\odot 0)$ is ϕ_2 fixed we have

$$\begin{array}{rcl} \Theta(F\odot 0) & = & \forall_{\pi_2}(a_{R^{op}}\vee\Theta(F\odot 0)\odot 0) \\ & \leq & \forall_{\pi_2}(a_{R^{op}}\vee a_R\vee\Theta(F\odot 0)\odot 0) \end{array}$$

and since $\Theta(0\odot a)$ is ϕ_1 fixed we similarly have

$$\begin{array}{rcl} \Theta(0\odot a) & = & \forall_{\pi_2}(a_R\vee\Theta(0\odot a)\odot 0) \\ & \leq & \forall_{\pi_2}(a_{R^{op}}\vee a_R\vee\Theta(0\odot a)\odot 0). \end{array}$$

By taking the meet of these two inequalities one obtains

$$\begin{array}{rcl} \Theta(F\odot 0)\wedge\Theta(0\odot a) & \leq & \forall_{\pi_2}(a_{R^{op}}\vee a_R\vee[\Theta(F\odot 0)\wedge\Theta(0\odot a)]\odot 0) \\ & \leq & \forall_{\pi_2}(a_{R^{op}}\vee a_R\vee N\odot 0) \end{array}$$

and so have finally completed the proof. ■

This proof provides yet another description of localic patch. All of them follow essentially the same line but all have distinct 'carrier' sets into which the frame of opens of the patch is embedded.

Paper	*Patch embeds into ...*
[BB88]	Frame of nuclei (see II 2.1 [J82] for the defintion)
[T96]	the ideal completion of the free Boolean algebra on ΩX (equivalently, the frame of distributive lattice congruences on ΩX),
[E01]	Frame of Scott continuous (i.e. directed join preserving) nuclei
here	$\mathbf{PreFr}(\Lambda\Omega X, \Lambda\Omega X)$

The previous section indicates that we could have also chosen $\mathbf{PreFr}(\Omega X, \Omega X)$ or indeed $\Lambda\Omega X \otimes_{\mathbf{PreFr}} \Omega X$ and this last choice may have eased the proof of compactness, using binary Tychonoff, but possibly introduced further algebraic complications. The important point of this proposition is that it shows that it is legitimate to *define* a stably locally compact locale to be one whose frame of opens is of the form $\Omega\overline{Y}$ for some compact Hausdorff localic poset. With the proposition this definition is seen to be equivalent to the usual lattice-theoretic one. It is with this definition that we can restate the main result:

Theorem 19 *(i) There is a bijection between ideal completion locales and discrete posets.*

(ii) There is a bijection between stably locally compact locales and ordered compact Hausdorff locales.

These two are now the same result and can be proved using the same method under the preframe/suplattice parallel. [T05] indicates how to express the parallel as a formal categorical order duality.

We end this section with an application of Escardó's description of patch. In contrast to [BB88] and [T96], [E01] gives an explicit description of the patch locale. Its frame of opens are exactly the Scott continuous nuclei; so the analogous observation for our work is that $Q = Im\Theta$. This is remarkable since it shows,

Theorem 20 *(i) If $R \hookrightarrow X \times X$ is a closed relation on an ordered compact Hausdorff locale such that*

(a) $R \leq \leq_X$

(b) $R; R = R$

(c) $\leq_X; R; \leq_X = R$

Then $R = \leq_X; (R \wedge \Delta); \leq_X$.

(ii) There exists $R \hookrightarrow X \times X$, a relation on a poset, such that (a), (b) and (c) of (i) hold, but $R \neq \leq_X; (R \cap \Delta); \leq_X$.

Spatially (i) is saying that if xRy then there is a z such $x \leq_X zRz \leq_X y$, which in the absence of topology is clearly not generally the case from the weak assumptions (a), (b) and (c). Part (ii) provides the counter-example.

Proof. (i) Using (c) and Theorem 7 such R correspond to preframe endomorphisms on the stably locally compact locale formed by the fixed points of $\Uparrow^{op}$. (a) and (b) are asserting that the preframe endomorphism corresponding to R is in fact a nucleus. It is Scott continuous since the morphism is a preframe homomorphism and so preserves directed joins. Since [E01] shows that all such nuclei are in the patch the conclusion is immediate from the spatial description of patch given in the previous section.

(ii) Take $X = \mathbb{Q}$ the rationals with its usual ordering and take $R = <$, the strictly less than relation. Note that $R \cap \Delta$ is the empty set. ■

Thus an application of Escardó's description of patch is that it offers a concrete example of a statement about compact Hausdorff locales that is expressible within the regular fragment of logic, but which is not mirrored by discrete locales.

7 Final Comments

It is satisfying that the duality between discrete and compact Hausdorff leads to an understanding that the patch construction and the representation of algebraic dcpos by posets are both the same abstract functor and can lead axiomatically to the same set of categorical equivalences (though we have omitted a discussion of morphisms). Along the way (a) the spatial nature of the patch construction, as an action on topologies, has been clarified and (b) some insight has been given into why a number of different constructions of patch occur.

Change of base arguments could have been deployed throughout and in some ways this may have been a better way to proceed. The relevant abstract approach would then be exactly that of [T05] together with an assumption that all the axioms are slice stable. Then for any discrete X there is an order isomorphism between maps $X \to \mathbb{S}^Y$ and internal join semilattice homomorphisms $\mathbb{S}^X \to \mathbb{S}^Y$ where $\mathbb{S}$ is the Sierpiński locale. This is a corollary to the Hofmann-Mislove theorem applied to X_Y; that is carried out relative to the slice category $\mathbf{Loc}/Y$ applied to the object $\pi_1 : Y \times X \to Y$. Together with the naturality of this observation our lemma 5 can be recovered. Extending, join semilattice homomorphisms $\mathbb{S}^{Idl(X)} \to \mathbb{S}^Y$ are in bijection with monotone maps $X^{op} \to \mathbb{S}^Y$ if X is further assumed to be a poset. To prove the main result all that needs to be verified is that in the slice $\mathbf{Loc}/Y$ maps $X_Y \times X_Y^{op} \to \mathbb{S}_Y$ that classify opens R enjoying $R = \leq; (R \wedge \Delta); \leq$ are in bijection with arbitrary opens $a_I : X_Y \to \mathbb{S}_Y$. This can be proved using only the regular fragment of logic and since the constructions of regular logic can be seen to be stable under change of base (and certainly valid in all slices) the main result is immediate. See [T03] for more details on how change of base works using categorical axioms. This approach makes the spatial nature of the argument clearer. However the path we chose above was to translate the spatial intuitions into statements about suplattice/preframe homomorphisms and then argue about these homomorphisms. The principal benefit is that we have not had to discuss change of base above and so have avoided the need to invoke Joyal and Tierney's result that the category of locales is slice stable (e.g. Theorem C1.6.3 [J02]).

Given that a poset has been central to the main result it is not clear how to extend this construction to the representation of continuous posets using continuous information systems (e.g. [V93]). Further given that a coherent locale has, as its opens, the ideal completion of a discrete distributive lattice, specializing the representation of stably locally compact locales to coherent locales (i.e. recovering localic Priestley duality) appears to require a mixture of both the discrete and compact Hausdorff sides of the parallel. This also remains as further work.

Finally it must be clarified that an intrinsic definition of stably locally compact would give the representation theorem more weight (rather than defining stably locally compact to be the 'preframe parallel' to ideal completion). Getting such an intrinsic definition, that fits into the parallel, is left as further work. The usual intrinsic definition (as a finitary meet stable continuous poset) does not seem to be available as the way below relation does not appear to work well under the parallel between preframe/suplattice. Other possible definitions could be as an injective object with respect to a class of monomorphisms or as a locale with the property that it is exponentiable and its upper power locale is an internal join semilattice. Thus this paper should be viewed as a first step: we have offered a spatial account of how the patch construction works as an action on topologies, but would like a better understanding of how the resulting representation theorems are indeed order dual.

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