THE PATCH CONSTRUCTION IS THE SAME THING AS ALGEBRAIC DCPO REPRESENTATION

Christopher Townsend

TALK OUTLINE.

A. DESCRIBE ALGEBRAIC DCPO REPRESENTATION TOPOLOGICALLY.

B. RECALL PATCH CONSTRUCTION.

C. TOPOLOGICAL PRODUCT $X \times Y$

* VIA SUPLATTICE TENSOR
* VIA PRE FRAME TENSOR

D. RELATIONAL COMPOSITION AS

* SUPLATTICE HOMOMORPHISM
* PREFRAME HOMOMORPHISM

E. ALGEBRAIC DCPO REPRESENTATION VIA SUPLATTICE HOMS.

PAINFUL BUT...

F. DUAL ARGUMENT VIA PREFRAME HOMS. GIVES PATCH

G. CATEGORICAL ACCOUNT.

H. FURTHER WORK: AXIOMATIC ACCOUNT OF STABLY LOCALLY COMPACT.

$$
\begin{tikzcd}
\text{Poset} \arrow[rr] & \simeq & \text{Algebraic DCPOs} \arrow[ll, dashed, bend right, "\text{Set of compact elements}"'] \\
(X, \leq) \arrow[rr, mapsto] & & \mathrm{idl}(X) \quad \text{Ideal completion}
\end{tikzcd}
$$

Topologically

$$
\begin{tikzcd}
\boxed{\text{Discrete Top. Poset}} \arrow[rr, "\text{Set of upper closed open}"] & \simeq & \boxed{\text{Topologies of the form } \mathcal{U}(X)} \arrow[ll, dashed, bend left, "\text{Next slide}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\boxed{\text{Compact Haus. Poset } (\leq \text{ closed})} \arrow[rr, "\text{Set of lower clsd opens}"] & \simeq [G80] & \boxed{\text{(Sober) stably locally compact spaces. } \Omega\overline{X}} \arrow[ll, dashed, bend left, "\underline{\text{Patch}}"]
\end{tikzcd}
$$

$\cup\!|$ $\qquad\qquad\qquad\qquad$ $\cup\!|$

<u>Ordered Stone Spaces</u> $\simeq$ <u>Coherent Spaces</u>

$\subseteq$ Stone

$$
\begin{tikzcd}
\text{Ordered Stone Spaces} \arrow[r, leftrightarrow, "\text{Priestley Duality}"] & \underline{\text{DLat}}^{op}
\end{tikzcd}
$$

GIVEN A POSET $(X, \leq)$ CAN WE RECOVER $P(X)$ FROM $\mathcal{U}(X)$?

WELL

$$PX \hookrightarrow \mathcal{U}(X^{op} \times X)$$

$$I \longmapsto \bigcup_{i \in I} \downarrow i \times \uparrow i$$

IS AN EMBEDDING.

[SINCE $i \in I \Rightarrow (i,i) \in \bigcup_{i \in I} \downarrow i \times \uparrow i = \bigcup_{j \in J} \downarrow j \times \uparrow j$
SO $i \leq j$ AND $j \leq i$ SOME $j \in J \Rightarrow i \in J$ ETC.]

WE CAN IN FACT RECOVER PX AS AN EQUALIZER

$$PX \hookrightarrow \mathcal{U}(X^{op} \times X) \underset{\Phi}{\overset{\mathrm{Id}}{\rightrightarrows}} \mathcal{U}(X^{op} \times X)$$

WHERE Φ IS IDEMPOTENT.

NEW FOR TODAY.

(1) CAN CONSTRUCT $\mathcal{U}(X^{op} \times X)$ AND Φ (AND HENCE PX) ONLY BY REFERENCE TO $\mathcal{U}(X)$.

(2) EXACTLY THE SAME CONSTRUCTION WORKS ON $\Omega \overline{X}$ THE OPENS OF A STABLY LOCALLY COMPACT $\overline{X}$, GIVING PATCH.

SPLITTING IDEMPOTENTS.

IF $\alpha : A \to A$ HAS $\alpha^2 = \alpha$ THEN IT SPLITS IF $\exists\ i : B \to A$ AND $r : A \to B$ WITH $ir = \alpha$ $ri = \mathrm{Id}_B$.

THIS MAKES B BOTH AN EQUALIZER (INCLUSION) AND COEQUALIZER (QUOTIENT): THAT IS $\forall C, C'$

$$\begin{tikzcd}
B \arrow[r, hook, "i"] & A \arrow[r, shift left, "\mathrm{Id}"] \arrow[r, shift right, "\alpha"'] & A \arrow[r, "r"] \arrow[dr, "c"'] & B \arrow[d, dashed, "\exists!"] \\
C' \arrow[u, dashed, "\exists!"] \arrow[ur, "c'"] & & & C
\end{tikzcd}$$

IF $\alpha c' = c'$ AND $c\alpha = c$.

EXAMPLE. $(X, \leq)$ A POSET. $\mathbb{I}$

$$\mathcal{U}(X^{op} \times X) \hookrightarrow P(X \times X) \xrightarrow{\Delta^{-1}} PX \xrightarrow{\exists_\Delta} P(X \times X) \xrightarrow{\downarrow \times \uparrow} \mathcal{U}(X^{op} \times X)$$

$$R \longmapsto \leq ; (R \cap \Delta) ; \leq$$

IS IDEMPOTENT.

PX SPLITS SINCE

$$\underbrace{\Delta^{-1}}_{r} \underbrace{(\downarrow \times \uparrow)\exists_\Delta}_{i}(I) = \left\{ j \,\middle|\, (j,j) \in \bigcup_{i \in I} \downarrow i \times \uparrow i \right\} = I$$

SO $ri = \mathrm{Id}_{PX}$.

GIVEN SPACES X, Y, DESCRIBE $X \times Y$.

EITHER SET $U \otimes V = \{(u,v) \mid u \in U, v \in V\}$

ALL $U \in \Omega X$ $V \in \Omega Y$ AS GENERATORS

THEN

$$\Omega(X \times Y) \cong \Omega X \otimes \Omega Y$$ SUP-LATTICE $\otimes$.

$\bigvee$ PRESERVING.

THEN FOR ALL SPACES Z,

$$\mathrm{Sup}(\Omega X \otimes \Omega Y, \Omega Z) \cong \mathrm{Sup}(\Omega X, \mathrm{Sup}(\Omega Y, \Omega Z))$$

OR SET $U \odot V = \{(u,v) \mid u \in U \text{ OR } v \in V\}$

$(= (U^c \otimes V^c)^c$ I.E DUAL TO $\otimes)$

THEN ON THESE GENERATORS...

$$\Omega(X \times Y) \cong \Omega X \otimes_{pf} \Omega Y$$ PREFRAME TENSOR

$\bigvee^{\uparrow}$ $\wedge$ PRESERVING

SO, $\forall$ SPACES Z,

$$\mathrm{PreFr}(\Omega X \otimes_{pf} \Omega Y, \Omega Z) \cong \mathrm{PreFr}(\Omega X, \mathrm{PreF}(\Omega Y, \Omega Z))$$

TECHNICAL LEMMAS.

Lemma 1: FOR DISCRETE X, Y

$$\Omega(X \times Y) \cong \mathrm{Sup}(\Omega X, \Omega Y)$$

AND RELATIONAL COMPOSITION MAPS TO FUNCTION COMPOSITION.

Lemma 2: FOR COMPACT HAUSDORFF X, Y

$$\Omega(X \times Y) \cong \mathrm{PreFr}(\Omega X, \Omega Y)$$

AND REL. COMPOSITION MAPS TO FUNCTION COMP.

PROOF 1

$$R \subseteq X \times Y \longmapsto (a \longmapsto a;R)$$

$$(1 \otimes \psi)(\Delta) \longleftarrow\!\shortmid\ \psi$$

PROOF 2

$$R^c \subseteq X \times Y \longmapsto (a \longmapsto (a^c;R)^c)$$

$$(1 \odot \psi)(\Delta^c) \longleftarrow\!\shortmid\ \psi$$

APPLICATION CREATING THE OPENS OF ALGEBRAIC DCPOS.

$(X,\leq)$ POSET

$$\begin{array}{ccc} \mathcal{P}(X\times X) & \xrightarrow{\cong} & \mathrm{SUP}(\mathcal{P}X,\mathcal{P}X) \\ \leq & \longmapsto & \uparrow \end{array}$$

$\mathcal{U}X$ IS THE SPLIT OF THE IDEMPOTENT $\uparrow$. //

APPLICATION GIVEN POSETS $(X,\leq)$ AND $(Y,\leq)$

$$\mathcal{U}(X^{op}\times Y) \cong \mathrm{Sup}(\mathcal{U}X,\mathcal{U}Y)$$

PROOF

$$\mathcal{P}X \underset{\uparrow}{\overset{\mathrm{id}}{\rightrightarrows}} \mathcal{P}X \longrightarrow \mathcal{U}X \cdots\cdots\to \mathcal{U}Y \hookrightarrow \mathcal{P}Y \underset{\uparrow}{\overset{\mathrm{id}}{\rightrightarrows}} \mathcal{P}Y$$

So
$$\begin{aligned} \mathrm{Sup}(\mathcal{U}X,\mathcal{U}Y) &\cong \{\bar{\Psi}:\mathcal{P}X\to\mathcal{U}Y \mid \bar{\Psi}\uparrow=\bar{\Psi}\} \\ &\cong \{\psi:\mathcal{P}X\to\mathcal{P}Y \mid \psi\uparrow=\psi,\ \uparrow\psi=\psi\} \\ &\cong \{R\subseteq X\times Y \mid \leq;R=R=R;\leq\} \\ &\cong \mathcal{U}(X^{op}\times Y) \end{aligned}$$
//

COROLLARY

$\mathrm{Sup}(\mathcal{U}X,\Omega)\cong\mathcal{U}(X^{op})$ $(Y=1)$

So $ev:\mathcal{U}X^{op}\otimes\mathcal{U}(X)\longrightarrow\Omega$

$$ev(\bar{I}\otimes I)=1 \iff \exists k\in\bar{I}\cap I.$$
//

COROLLARY $\mathcal{U}(X^{op})\otimes\mathcal{U}(X)\cong\mathcal{U}(X^{op}\times X)$ //

Finally: How to Recover PX from $\mathcal{U}X$.

Recall

$$\begin{tikzcd} PX \arrow[r, hook] & \mathcal{U}(X^{op}\times X) \arrow[r, shift left, "id"] \arrow[r, shift right, "\leq;(-\cap\Delta);\leq"'] \arrow[d, phantom, "\cong" rotate=90] & \mathcal{U}(X^{op}\times X) \arrow[d, phantom, "\cong" rotate=90] \\ & \mathcal{U}(X^{op})\otimes\mathcal{U}(X) \arrow[r, shift left, "id"] \arrow[r, shift right, "\Phi"'] & SUP(\mathcal{U}X,\mathcal{U}X) \end{tikzcd}$$

Transpose:

$$\hat{\Phi} : \mathcal{U}(X^{op})\otimes\mathcal{U}(X)\otimes\mathcal{U}(X) \longrightarrow \mathcal{U}(X) \cong SUP(\mathcal{U}X^{op},\Omega)$$

Again

$$\hat{\hat{\Phi}} : \mathcal{U}(X^{op})\otimes\mathcal{U}(X)\otimes\mathcal{U}(X)\otimes\mathcal{U}(X^{op}) \longrightarrow \Omega$$

How About $\hat{\hat{\Phi}}(\bar{I}\otimes I\otimes J\otimes\bar{J}) = ev(\bar{I}\cap\bar{J}\otimes I\cap J)$?

Yes! $R_{\leq;(-\cap\Delta);\leq} \subseteq X\times X\times X\times X$ has

$$\chi_R(i,j,\bar{i},\bar{j}) = 1 \iff (\bar{i},\bar{j}) \in \leq;(\downarrow i\times\uparrow j\cap\Delta);\leq$$

$$\iff \exists k \quad \bar{i}\leq k\leq\bar{j} \quad j\leq k\leq i.$$

But $ev(\downarrow i\cap\downarrow\bar{j}\otimes\uparrow j\cap\uparrow\bar{i}) = 1 \iff \exists k\in\downarrow i\cap\downarrow\bar{j}\cap\uparrow j\cap\uparrow\bar{i}$ //

$\boxed{\text{So}}$

$$PX \hookrightarrow SUP(\mathcal{U}X,\Omega)\otimes\mathcal{U}X \overset{id}{\underset{\Phi}{\rightrightarrows}} SUP(\mathcal{U}X,\mathcal{U}X)$$

where Φ is the transpose of an evaluation map.

THE SAME THING FOR COMPACT HAUS. $(X, \leq)$

$$\Omega(X \times X) \cong \mathrm{PreFr}(\Omega X, \Omega X)$$
$$\leq^c \longmapsto \Uparrow^{op}$$

$\Omega \overline{X}$ IS (DEFINITION) SPLITTING OF $\Uparrow^{op}$.

FOR $(X, \leq), (Y, \leq)$ $\Omega(\overline{X^{op} \times Y}) \cong \mathrm{PreFr}(\Omega \overline{X}, \Omega \overline{Y})$

SO

* $\mathrm{PreFr}(\Omega \overline{X}, \Omega) \cong \Omega \overline{X}^{op}$ $(Y=1)$
* $ev: \Omega \overline{X}^{op} \otimes_{pf} \Omega \overline{X} \longrightarrow \Omega$
* $\Omega \overline{X^{op}} \otimes_{pf} \Omega \overline{X} \cong \Omega(\overline{X^{op} \times X})$

AGAIN THE ACTION $\Omega(\overline{X^{op} \times X}) \xrightarrow{\mathbb{I}} \Omega(\overline{X^{op} \times X})$

$$R^c \longmapsto (\leq ; R \cap \Delta ; \leq)^c$$

IS TRANSPOSE OF

$$\Omega(\overline{X^{op}}) \otimes_{pf} \Omega \overline{X} \otimes_{pf} \Omega \overline{X} \otimes_{pf} \Omega(\overline{X^{op}}) \longrightarrow \Omega$$

$$\bar{I} \odot I \odot J \odot \bar{J} \longmapsto ev(\bar{I} \vee \bar{J} \odot I \vee J)$$

JUST AS IN DISCRETE CASE LEFT CHECKING...

$$\Omega(\overline{X^{op} \times X}) \hookrightarrow \Omega X \otimes_{pf} \Omega X \xrightarrow{\Delta^{-1}} \Omega X \xrightarrow{\forall_\Delta} \Omega(X \times X) \xrightarrow{\Downarrow^{op} \odot \Uparrow^{op}} \Omega(\overline{X^{op} \times X})$$

SPLITS TO ΩX.

I.E. $\Delta^{-1} \Downarrow^{op} \odot \Uparrow^{op} \forall_\Delta(U) = U$ ANY OPEN $U \in \Omega X$.

TAKE COMPLEMENTS THIS IS: $(\leq ; (\exists_\Delta C) ; \leq) \cap \Delta = C$

ALL CLOSED C. //

... OR **ESCARDÓ**

DO I REALLY MEAN THE SAME?

OBJECTIONS: COMPACT HAUSDORFF $\neq$ DISCRETE.

PREFRAME $\neq$ SUPLATTICE.

$\boxed{\text{YES.}}$ WE CAN AXIOMATIZE A FRAGMENT OF TOPOLOGY AS AN ORDER ENRICHED CATEGORY $\mathbb{C}$. THEN, WITH RESPECT TO THIS AXIOMATIZATION,

$$X \in \mathbb{C} \text{ COMPACT HAUSDORFF}$$

$$\iff X \in \mathbb{C}^{co} \text{ IS DISCRETE.}$$

THERE IS A FUNCTOR PATCH

$$\underline{\text{StLoc}}_{\mathbb{C}} \longrightarrow \underline{\text{KHausPos}}_{\mathbb{C}}$$

WHOSE ORDER DUAL IS ALG. DCPO REP.

$$\underline{\text{Alg-DCPO}}_{\mathbb{C}} \longrightarrow \underline{\text{Pos}}_{\mathbb{C}} .$$

Flavour of This Axiomatization.

* $\mathbb{C}$ has finite products
* $\mathbb{C}$ order enriched
* There is $\$ \in \mathbb{C}$ order internal distributive lattice.

$$(\Delta \dashv \sqcap \; , \; \sqcup \dashv \Delta)$$

Then $\mathbb{C} = \underline{\mathrm{Top}}$

$$\underline{\mathrm{Top}}(\$^X, \$^Y) \cong \underline{\mathrm{dcpo}}(\Omega X, \Omega Y)$$

$$\begin{tikzcd}
\sqcap\text{-SLAT}(\$^X, \$^Y) \arrow[r, bend left, "(\ )^{co}"] \arrow[d, phantom, "\cong"] & \sqcup\text{-SLAT}(\$^X, \$^Y) \arrow[l, bend left, "(\ )^{co}"] \arrow[d, phantom, "\cong"] \\
\text{PreFrame} & \text{Sup-Lattice Homs.}
\end{tikzcd}$$

TOPOLOGY?

RECALL X COMPACT $\Leftrightarrow$ $\Omega ! : \Omega \longrightarrow \Omega X$ HAS A PREFRAME RIGHT ADJ.

AXIOMATICALLY $X \in \mathbb{C}$ COMPACT $\Leftrightarrow$ $\$^! : \$ \to \$^X$ HAS A RIGHT ADJOINT.

SIMILARLY (VERMEULEN) X IS COMPACT HAUSDORFF $\Leftrightarrow$ (COMPACT &) $\Omega\Delta : \Omega(X \times X) \longrightarrow \Omega X$ HAS PREFRAME RIGHT ADJOINT AND

$$\forall_\Delta (a \vee \Omega\Delta(I)) = \forall_\Delta a \vee I.$$

DUALLY X IS DISCRETE $\Leftrightarrow$ $\Omega\Delta$ HAS LEFT ADJOINT $\exists_\Delta : \Omega X \longrightarrow \Omega(X \times X)$ WITH

$$\exists_\Delta (a \wedge \Omega\Delta(I)) = \exists_\Delta a \wedge I$$

(JOYAL & TIERNEY).

$\Rightarrow$ ORDER DUAL CONCEPTS.

FURTHER WORK.

WANT AN INTRINSIC DEFINITION OF STABLY LOCALLY COMPACT.

IN FACT $\overline{X}$ STABLY LOCALLY COMPACT

$\Leftrightarrow$ $\Omega\overline{X}$ CONTINUOUS POSET

$$(a = \bigvee^{\uparrow} \{ b \mid b \ll a \})$$

AND $1 \ll 1$ AND $a \ll b_1, b_2 \Rightarrow a \ll b_1 \wedge b_2$

BUT OUR "DEFINITION" HAS BEEN

$\Leftrightarrow$ $\Omega\overline{X}$ IS A SPLITTING OF $\Omega X \underset{\alpha_{\leq}}{\overset{\mathrm{id}}{\rightrightarrows}} \Omega X$

FOR KHAUS $(X, \leq)$.

THIS MIRRORS THE DEFINITION OF (THE OPENS OF) ALGEBRAIC DCPOs.

THIS ENSURES THE FUNCTOR $(X, \leq) \mapsto \overline{X}$ IS ESSENTIAL SURJECTIVE BY DEFINITION.

NICER TO HAVE AN INTRINSIC DEFINITION

OF (i) STABLY LOCALLY COMPACT

AND DUALLY (ii) ALGEBRAIC DCPO.

SUMMARY

* ALGEBRAIC DCPO REPRESENTATION CAN BE VIEWED AS AN ACTION ON TOPOLOGIES.

* VIEWED AS SUCH IT IS THE SAME CONSTRUCTION AS PATCH.

* THERE IS A DUALITY BETWEEN SUPLATTICE HOMS AND PREFRAME HOMS.

* THE KEY IS TO REPRESENT RELATIONS ON DISCRETE SPACES AS SUPLATTICE HOMS & CLOSED RELATIONS ON COMP. HAUS. SPACES AS PREFRAME HOMS.

* THE DUALITY CAN BE MADE FORMAL BY STATING THE RESULTS RELATIVE TO A SUITABLY AXIOMATIZED ORDER ENRICHED CATEGORY $\mathbb{C}$.