

(1)

TOWARDS A LOCALIC PROOF
OF THE LOCALIC GROUPOID
REPRESENTATION OF GROTHENDIECK
TOPOSES.

CHRISTOPHER TOWNSEND
SEPT. 2011

OUTLINE

1. BACKGROUND/NOTATION.
2. THE REPRESENTATION THEOREM & PROOF.
3. WHAT IS A "LOCALIC PROOF"?
4. LOCALIC PROOF STRATEGY.

1. BACKGROUND/NOTATION

$\mathcal{F}, \mathcal{E}, \mathcal{S}$ - ELEMENTARY TOPOSES.

$f: \mathcal{F} \rightarrow \mathcal{E}, p: \mathcal{E} \rightarrow \mathcal{S}$ - GEOMETRIC MORPHISMS.

$\underline{\text{Loc}}_{\mathcal{E}}, \underline{\text{Loc}}_{\mathcal{S}}$ - CATEGORIES OF LOCALES (OVER \mathcal{E}, \mathcal{S})

$p: \mathcal{E} \rightarrow \mathcal{S}$ IS LOCALIC IF $\forall A \in \text{Ob}(\mathcal{E})$ THERE IS A DIAGRAM

$$\begin{array}{ccc} \bar{A} & \longrightarrow & A \\ & \downarrow & \\ p^*I & & \end{array}$$

EQUIVALENTLY: $\exists X$ A LOCALE OVER \mathcal{S} , $\mathcal{E} \simeq \text{Sh}_{\mathcal{S}}(X)$

(VIA $p^* \dashv (p_*)$).

LOCALIC GEOMETRIC MORPHISMS ARE STABLE UNDER PULLBACK.

NOTE:

$$\begin{array}{ccc} \bullet & \xrightarrow{g} & \bullet \\ & \searrow fg & \downarrow f \\ & & \bullet \end{array}$$

fg LOCALIC $\Rightarrow g$ LOCALIC.

$p: \mathcal{E} \rightarrow \mathcal{S}$ IS BOUNDED IF $\exists B \in \text{Ob}(\mathcal{E})$, THE 'BOUND', SUCH THAT $\forall A \in \text{Ob}(\mathcal{E})$ THERE IS A DIAGRAM:

$$\begin{array}{ccc} \bar{A} & \longrightarrow & A \\ & \downarrow & \\ p^*I \times B & & \end{array}$$

EQUIVALENTLY, $\exists (\mathcal{C}, \mathcal{J})$ A SITE OVER \mathcal{S} WITH $\mathcal{E} \simeq \text{Sh}_{\mathcal{S}}(\mathcal{C}, \mathcal{J})$

(VIA p) I.E. \mathcal{E} IS A GROTHENDIECK TOPOS OVER \mathcal{S} .

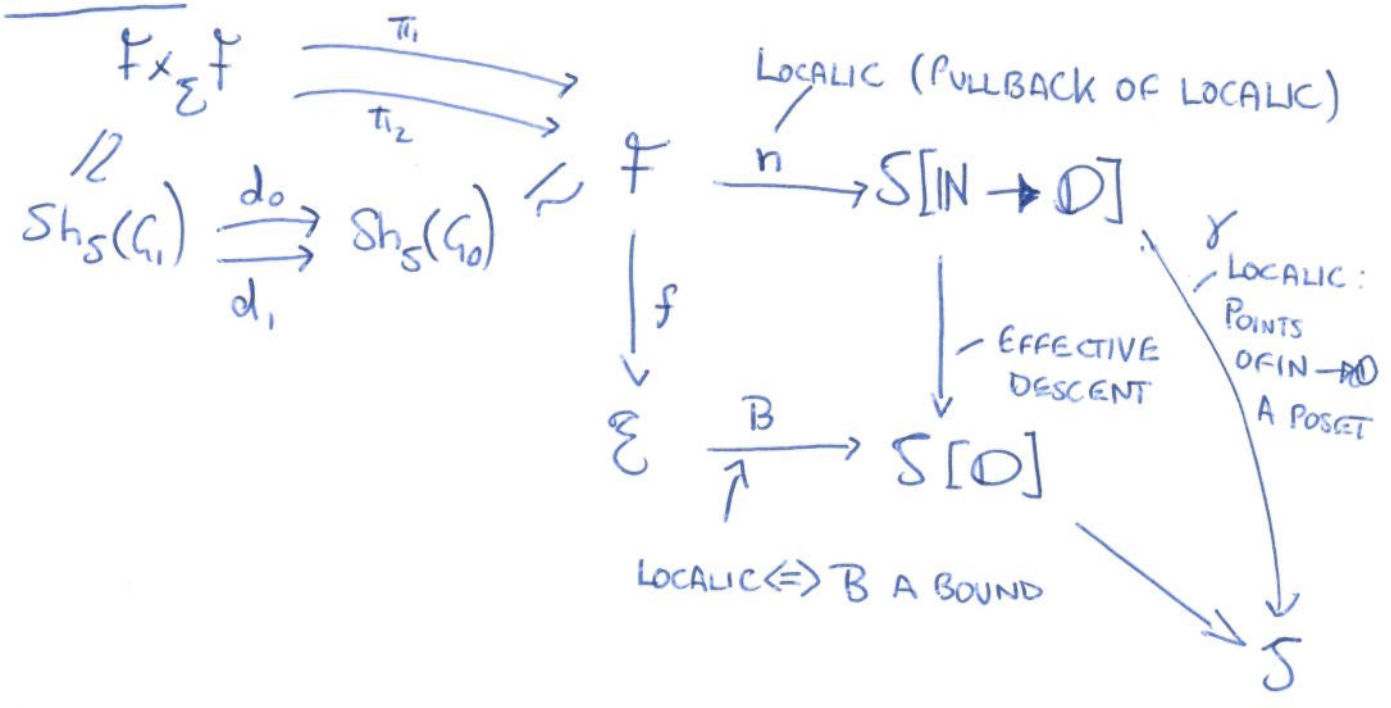
GIVEN $f: \mathcal{F} \rightarrow \mathcal{E}$ DEFINE $\text{DES}_f(\mathcal{F})$ TO HAVE OBJECTS
 $(A \in \mathcal{A}b(\mathcal{F}), \theta: \pi_1^* A \rightarrow \pi_2^* A)$ WHERE $\pi_i: \mathcal{F} \times_{\Sigma} \mathcal{F} \rightarrow \mathcal{F}$ AND
 θ SATISFIES UNIT & COCYCLE CONDITIONS. THERE IS A
 FUNCTOR $f^*: \Sigma \rightarrow \text{DES}_f(\mathcal{F})$ AND f IS AN EFFECTIVE
DESCENT MORPHISM IF f^* IS AN EQUIVALENCE

2. THE JOYAL & TIERNEY REPRESENTATION THEOREM.

THEOREM [JT] IF \mathcal{E} IS A GROTHENDIECK TOPOS OVER \mathcal{S} THEN
 THERE EXISTS \mathcal{G} A LOCALIC GROUPOID OVER \mathcal{S} SUCH THAT

$$\Sigma \simeq \mathcal{B}\mathcal{G} \equiv \left\{ (X \xrightarrow{1} \mathcal{G}_0, \mathcal{G}_1 \times_{\mathcal{G}_0} X \xrightarrow{\alpha} X) \mid \begin{array}{l} \downarrow \text{A LOCAL HOMEOMORPHISM} \\ \text{\×} \text{ AN ACTION} \end{array} \right.$$

PROOF OUTLINE

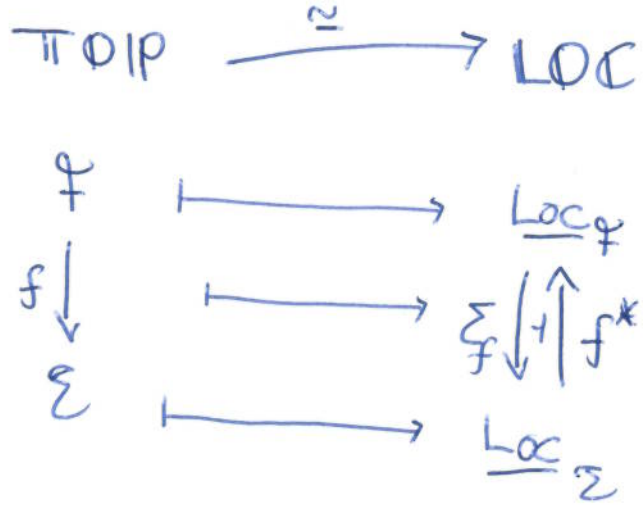


- f OF EFFECTIVE DESCENT (PULLBACK STABILITY)
- f_n LOCALIC $\Rightarrow \exists$ LOCALIC G_0 AS SHOWN
- f LOCALIC $\Rightarrow \pi_1$ LOCALIC $\Rightarrow \exists G_1$ AS SHOWN (& d_0, d_1 ETC)

SO $\Sigma \cong \text{DES}_f(\mathcal{F})$. OBJECTS OF $\text{DES}_f(\mathcal{F})$ ARE PAIRS $(A, \theta: \pi_1^*A \rightarrow \pi_2^*A)$ AND THESE CORRESPOND TO OBJECTS OF BG SINCE $\mathcal{F} \cong \text{Sh}_S(G_0)$ (AND OBJECTS OF $\text{Sh}_S(G_0)$ ARE LOCAL HOMOMORPHISMS OVER G_0). THE UNIT & COCYCLE CONDITIONS CORRESPOND TO ' α IS AN ACTION' IN THE DEFINITION OF BG .

□

3. WHAT'S A 'LOCALIC' PROOF?



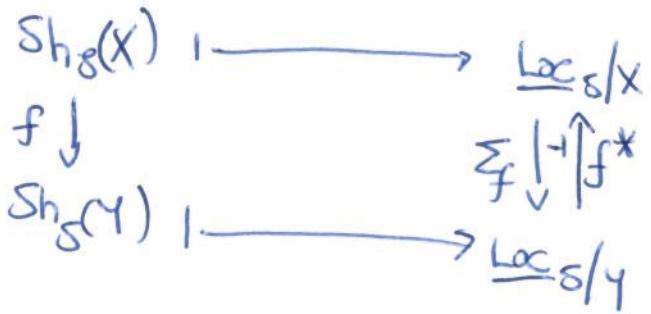
FROBENIUS RECIPROCALITY

$$\sum_f (W \times f^* X) \cong \sum_f W \times X$$

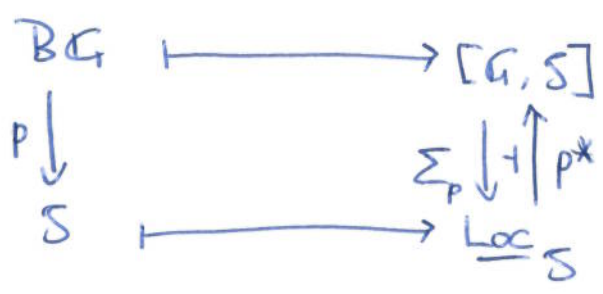
$$f^* \mathbb{L}OC_\Sigma \cong \mathbb{L}OC_f$$

ALLOWS A REPRESENTATION OF GEOMETRIC MORPHISMS. IDEA IS TO REASON WITH THESE 'LOCALIC' REPRESENTATIONS.

EXAMPLES



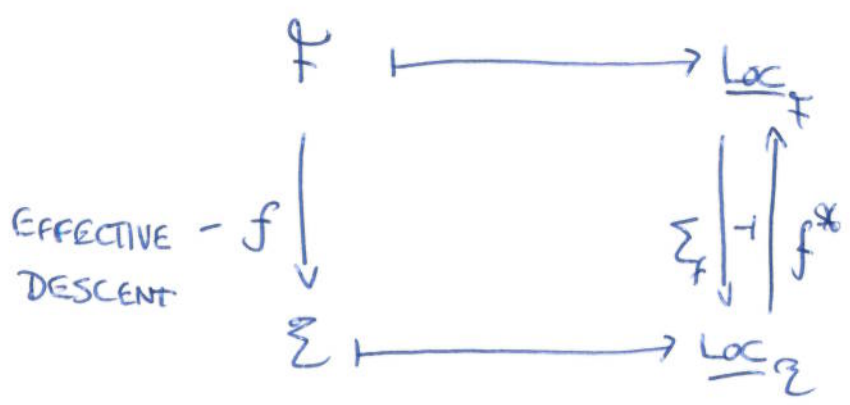
- USUAL PULLBACK ADJUNCTION



$\Sigma_p(X, X)$ GIVEN BY COEQUALIZER

$$G_1 \times_{G_0} X \xrightarrow[\pi_2]{\alpha} X \rightarrow \Sigma_p(X, X)$$

(NOTE: $\Sigma_p(G_1, m) \cong G_0$)



EFFECTIVE DESCENT

$\Leftrightarrow f^*$ MONADIC.

[JT] ESSENTIALLY REDUCES TO:

LEMMA: Σ A GROTHENDIECK TOPOS OVER S ($\Sigma \xrightarrow{p} S$) THEN THERE EXISTS \mathcal{G} , LOCALIC GROUPOID OVER S , SUCH THAT

$$\left(\begin{array}{c} \text{Loc}_{\Sigma} \\ \downarrow \text{p} \quad \uparrow \text{p} \\ \text{Loc}_S \end{array} \right) \cong \left(\begin{array}{c} [\mathcal{G}, S] \\ \downarrow \quad \uparrow \\ \text{Loc}_S \end{array} \right)$$

SAY "ESSENTIALLY" SINCE FROM $\text{Loc}_{\Sigma} \cong [\mathcal{G}, S]$ ONE STILL NEEDS TO VERIFY THAT THE EQUIVALENCE RESTRICTS TO DISCRETE LOCALES / LOCAL HOMEOMORPHISMS. HOWEVER THIS FOLLOWS EASILY FROM THE CONSTRUCTION OF \cong .

OUR "LOCALIC PROOF STRATEGY" STARTS WITH CHARACTERISING

THIS SITUATION:

LEMMA GIVEN $D \xrightleftharpoons[p^*]{\Sigma_f} C$, D, C CARTESIAN THEN $(D \xrightleftharpoons{\Sigma_f} C)$

IS EQUIVALENT TO $[\mathcal{G}, C] \xrightleftharpoons[p^*]{\Sigma_f} C$ FOR SOME GROUPOID \mathcal{G} IN C IFF $\exists W \rightarrow 1$ OF EFFECTIVE DESCENT (IE $W^*: D \rightarrow D/W$

MONADIC) SUCH THAT $D/W \xrightarrow{\Sigma_f} C/\Sigma_p W$ IS AN EQUIVALENCE

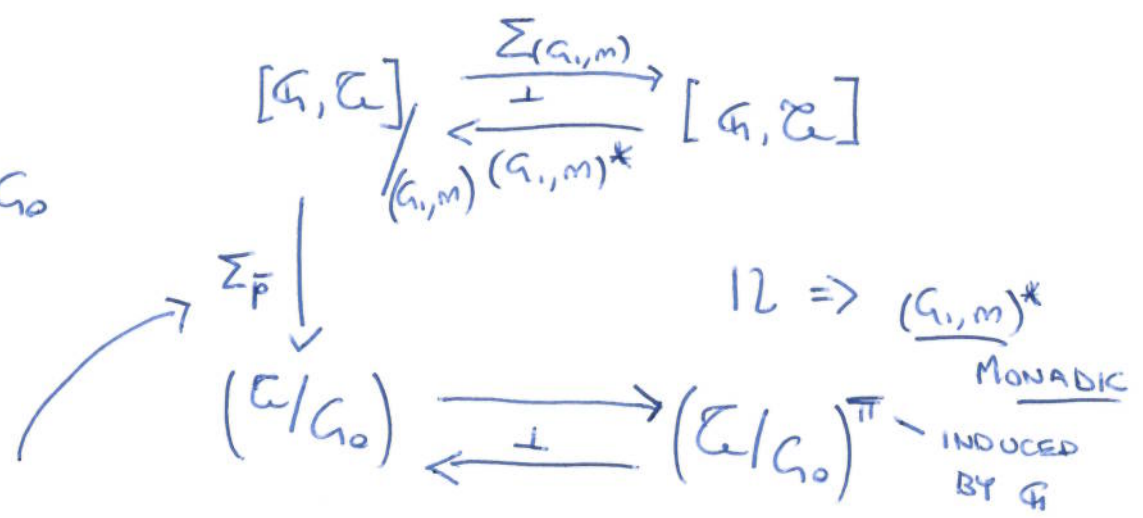
OF CATEGORIES, WHERE $\Sigma_p(W' \xrightarrow{f} W) \equiv \Sigma_p W' \xrightarrow{\Sigma_p f} \Sigma_p W$.

PROOF \Rightarrow (I.E. GIVEN $[G, \mathcal{C}] \xrightleftharpoons[\text{pt}]{\Sigma_p} \mathcal{C}$)

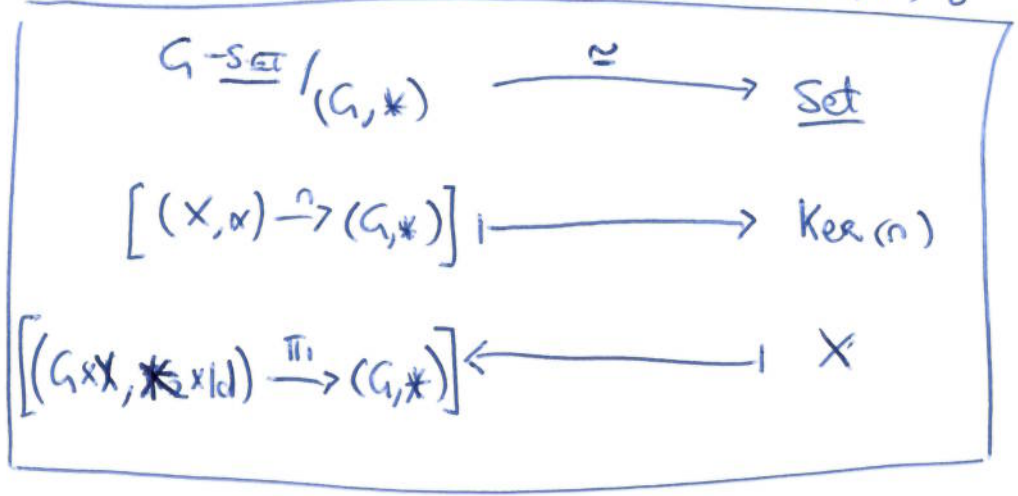
CONSIDER THE OBJECT (G, m) OF $[G, \mathcal{C}]$

RECALL:

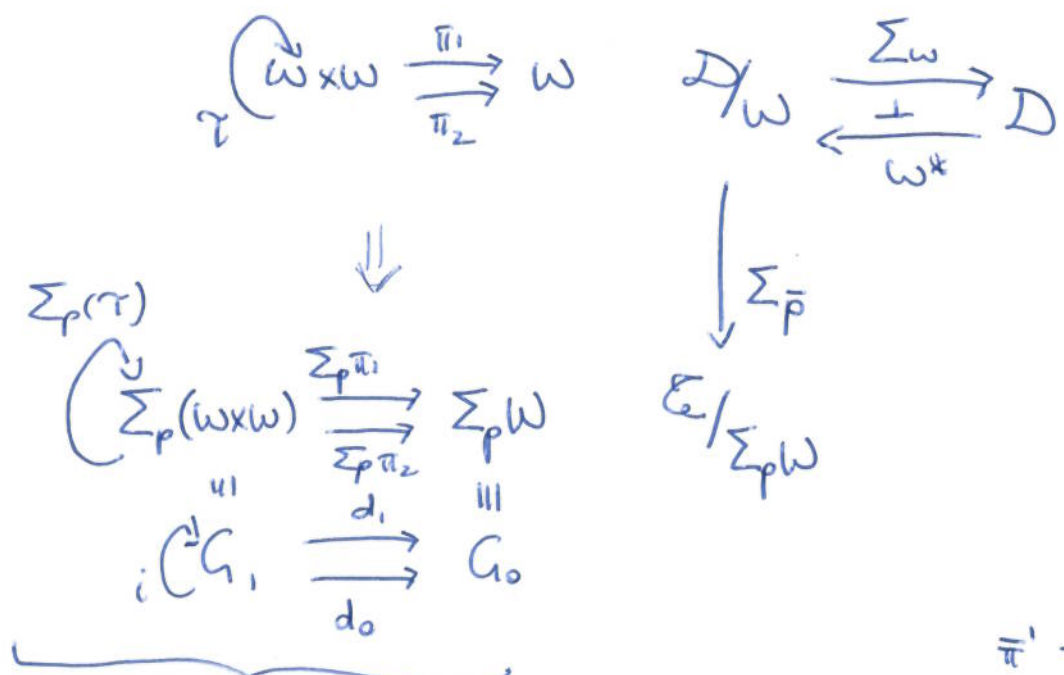
$\Sigma_p(G, m) \cong G_0$



PROVE THIS IS AN EQUIVALENCE JUST AS YOU WOULD PROVE, FOR ANY GROUP $(G, *)$:



PROOF CONT. \Leftarrow



A GROUPOID AS PER ~~BE~~ EARLIER EXAMPLE.

$\bar{\pi}$ THE 'IMAGE' OF π VIA THE EQUIVALENCE $\Sigma_{\bar{p}}$

$$\mathcal{D} \simeq (\mathcal{D}/\omega)^{\bar{\pi}} \xrightarrow[\simeq]{\text{INDUCED BY } \Sigma_{\omega^*} \omega^*} (\mathcal{E}/G_0)^{\bar{\pi}'} = [G, \mathcal{E}] \quad \square$$

THIS TECHNIQUE IS KNOWN; [SGA 4]: \mathcal{E} IS ÉTENDU IF $\exists B \rightarrow 1$ " \mathcal{E} S.T. $\mathcal{E}/B \simeq \text{Sh}_S(G_0)$. SO IF \mathcal{E} IS ÉTENDU THEN:

$$\begin{array}{ccc}
 \mathcal{E}/B \times B & \xrightarrow[\pi_2]{\pi_1} & \mathcal{E}/B \longrightarrow \mathcal{E} \\
 \parallel & & \parallel \\
 \text{Sh}(G_1) & \xrightarrow{\quad} & \text{Sh}(G_0)
 \end{array}$$

$$\Rightarrow \mathcal{E} \simeq B G$$

So BY THE LEMMA [JT] REDUCES TO: \exists GROTHENDIECK $\Rightarrow \exists W \rightarrow$
 OF EFFECTIVE DESCENT IN $\underline{\text{Loc}}_{\mathcal{E}}$ S.T. $\underline{\text{Loc}}_{\mathcal{E}/W} \xrightarrow{\Sigma_P} \underline{\text{Loc}}_{\mathcal{S}/\Sigma_P W}$
 IS AN EQUIVALENCE.

NOTE THAT BY OUR PROOF GIVEN ABOVE OF [JT] WE KNOW
 WHAT W 'SHOULD BE' IE TAKE $W = [IN \rightarrow B]$, THE
 LOCALE OF SURJECTIONS FROM IN TO B. I.E. THIS SHOWS US
 HOW TO GET W (AND SO \mathcal{R}) FROM THE BOUND B.

TO REDUCE FURTHER, WHAT IS THE LOCALIC INTERPETATION
 OF $\underline{\text{Loc}}_{\mathcal{E}/W} \xrightarrow{\Sigma_P} \underline{\text{Loc}}_{\mathcal{S}/\Sigma_P W}$? WELL THERE IS AN ADJUNCTIC

$$\underline{\text{Loc}}_{\mathcal{E}/W} \xrightarrow[\mathcal{Z}_W^*[p^*(-)]]{\Sigma_P} \underline{\text{Loc}}_{\mathcal{S}/\Sigma_P W} \quad - \textcircled{1}$$

IE RIGHT ADJOINT GIVEN BY -

$$\begin{array}{ccc} X & & \\ \downarrow \eta & \longrightarrow & \boxed{\begin{array}{ccc} \mathcal{Z}_W^* p^*(n) & \longrightarrow & p^* X \\ \downarrow & & \downarrow p^*(n) \\ W & \xrightarrow{\mathcal{Z}_W} & p^* \Sigma_P W \end{array}} \\ \Sigma_P W & & \end{array}$$

WHERE \mathcal{Z} IS UNIT OF $\Sigma_P \dashv p^*$. NOTE THAT UNIT OF (1), AT
 $f: Y \rightarrow W$ IS

$$Y \xrightarrow{(f, \mathcal{Z}_Y)} W \times_{p^* \Sigma_P W} p^* \Sigma_P Y$$

SO, CONCERNING $\Sigma_{\bar{p}}$, WE KNOW

(11)

① $\Sigma_{\bar{p}}$ IS PART OF AN ADJUNCTION: $\Sigma_{\bar{p}} \dashv \bar{p}^*$

② $\Sigma_{\bar{p}}(1) = 1$

③ $\Sigma_{\bar{p}} \dashv \bar{p}^*$ SATISFIES FROBENIUS RECIPROcity -
IT IS REPRESENTATION OF $\bar{p}: \text{Sh}_{\mathbb{Z}}(W) \rightarrow \text{Sh}_{\mathbb{Z}}(\mathbb{Z}_f)$

BUT CONSIDER THE LEMMA:

LEMMA: $L \dashv R: \mathcal{D} \rightleftarrows \mathcal{C}$ AN ADJUNCTION BETWEEN CARTESIAN CATEGORIES, WITH $L1 = 1$ AND $L \dashv R$ SATISFYING FROBENIUS RECIPROcity, THEN Σ REGULAR MONIC $\Rightarrow L$ IS AN EQUIV.

PROOF: $LRX \cong L(1 \times X) \cong L1 \times X \cong X \Rightarrow \Sigma$ ISO.

SO BY TRIANGULAR IDENTITIES $L\Sigma$ IS ISO. BUT THERE IS

AN EQUALIZER $W \xrightarrow{\Sigma W} RLW \begin{matrix} \xrightarrow{a_1} \\ \xrightarrow{a_2} \end{matrix} W' \Rightarrow La_1 = La_2$

SINCE $L\Sigma$ ISO. BUT THEN $a_1 = a_2$ AS $\Sigma W'$ MONO.

$\Rightarrow \Sigma W$ ISO $\Rightarrow L$ AN EQUIVALENCE \square

AND SO [JT] REDUCES TO THE FOLLOWING LEMMA:

LEMMA: \mathcal{E} A GROTHENDIECK TOPOS OVER \mathcal{S} (VIA $p: \mathcal{E} \rightarrow \mathcal{S}$)

THEN $\exists W \rightarrow 1$ OF EFFECTIVE DESCENT IN $\underline{\text{Loc}}_{\mathcal{E}}$ SUCH THAT

$\forall f: Y \rightarrow W \quad Y \xrightarrow{(f, \Sigma Y)} W \times_{p^* \Sigma_p W} p^* \Sigma_p Y$

IN PRACTICE WE WOULD GUESS TO TAKE $w = [N \rightarrow B]$. (12)

SO IN SUMMARY AN ENTIRELY LOCALIC ACCOUNT OF [JT]

CAN BE GIVEN IF WE CAN SHOW THE FOLLOWING TWO CONDITIONS ARE EQUIVALENT FOR ANY OBJECT B OF \mathcal{E} AND GEOMETRIC MORPHISM $p: \mathcal{Z} \rightarrow \mathcal{S}$.

(I)

$$\forall f: Y \rightarrow [N \rightarrow B]$$

$$\Omega[N \rightarrow B] \otimes_{\Omega p^* \Sigma_p Y} \xrightarrow{\Omega(f, \Sigma_Y)} \Omega Y \quad \text{A SURJECTION}$$

(II)

$$\forall A \in \text{Ob}(\mathcal{E})$$

$$B \times p^* p_*(\tilde{A}^B) \xrightarrow{\text{Id} \times \Sigma^{\mathcal{G}^m}} B \times \tilde{A}^B \xrightarrow{\text{ev}} \tilde{A} \quad \text{A SURJECTION}$$

FOR (I) RECALL THAT A LOCALE MAP $f: X \rightarrow Y$ IS A REG. MONIC IFF $\Omega f: \Omega Y \rightarrow \Omega X$ IS A SURJ.

FOR (II) CONSULT B3.1.6 OF JOHNSTONE'S 'ELEPHANT' FOR THE EASY PROOF THAT THIS IS EQUIVALENT TO THE CONDITION 'B IS A BOUND FOR p'. \tilde{A} IS THE PARTIAL MAP CLASSIFIER ON A AND $\Sigma^{\mathcal{G}^m}$ IS THE COUNT OF $p^* \dashv p_*$. $\Sigma^{\mathcal{G}^m}$

FACTORS VIA $\Omega \mathcal{Z}$, BUT THE EQUIVALENCE OF (I) AND

(II) HAS NOT BEEN CONFIRMED.