TOWARDS A LOCALIC PROOF OF THE LOCALIC GROUPOID REPRESENTATION OF GROTHENDIECK TOPOSES.

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SEPT. 2011

1. BACKGROUND/NOTATION

$\mathcal{F}, \mathcal{E}, \mathcal{S}$ - ELEMENTARY TOPOSES.

$f: \mathcal{F} \to \mathcal{E}$, $p: \mathcal{E} \longrightarrow \mathcal{S}$ - GEOMETRIC MORPHISMS.

$\underline{Loc}_{\mathcal{E}}$, $\underline{Loc}_{\mathcal{S}}$ - CATEGORIES OF LOCALES (OVER $\mathcal{E}, \mathcal{S}$)

$p : \mathcal{E} \to \mathcal{S}$ IS LOCALIC IF $\forall A \in Ob(\mathcal{E})$ THERE IS A DIAGRAM

$$\begin{tikzcd}
\bar{A} \arrow[r] \arrow[d, hook] & A \\
p^*I &
\end{tikzcd}$$

EQUIVALENTLY: $\exists X$ A LOCALE OVER $\mathcal{S}$, $\mathcal{E} \simeq Sh_{\mathcal{S}}(X)$ (VIA $p^* \dashv p_*$).

LOCALIC GEOMETRIC MORPHISMS ARE STABLE UNDER PULLBACK.

NOTE:

$$\begin{tikzcd}
\bullet \arrow[r, "g"] \arrow[dr, "fg"'] & \bullet \arrow[d, "f"] \\
& \bullet
\end{tikzcd}$$

fg LOCALIC $\Rightarrow g$ LOCALIC.

$p : \mathcal{E} \to \mathcal{S}$ IS BOUNDED IF $\exists B \in Ob(\mathcal{E})$, THE 'BOUND', SUCH THAT $\forall A \in Ob(\mathcal{E})$ THERE IS A DIAGRAM:

$$\begin{tikzcd}
\bar{A} \arrow[r] \arrow[d, hook] & A \\
p^*I \times B &
\end{tikzcd}$$

EQUIVALENTLY, $\exists (\mathbb{C}, J)$ A SITE OVER $\mathcal{S}$ WITH $\mathcal{E} \simeq Sh_{\mathcal{S}}(\mathbb{C}, J)$ (VIA p) IE $\mathcal{E}$ IS A GROTHENDIECK TOPOS OVER $\mathcal{S}$.

$\mathcal{C}$ CARTESIAN, THEN G A GROUPOID IN $\mathcal{C}$ IS THE DATA:

$$\begin{tikzcd} G_1 \times_{G_0} G_1 \arrow[r, "m"] & G_1 \arrow[loop below, "i"] \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & G_0 \arrow[l, bend right=60, "s"'] \end{tikzcd} \qquad ig = g^{-1}$$

AN INTERNAL CATEGORY, WITH ALL MORPHISMS ISOMORPHISM.

EXAMPLE 1: $\omega \in \mathcal{O}b(\mathcal{D})$, $\mathcal{D}$ CARTESIAN

$$\begin{tikzcd} \omega \times \omega \times \omega \arrow[r, "\pi_{13}"] & \omega \times \omega \arrow[loop below, "\tau"] \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \omega \arrow[l, bend right=60, "\Delta"'] \end{tikzcd}$$

EXAMPLE 2: FURTHER, GIVEN $F : \mathcal{D} \to \mathcal{C}$, $\mathcal{C}$ CARTESIAN SUCH THAT $F : \mathcal{D}/\omega \to \mathcal{C}/F\omega$ IS AN EQUIVALENCE:

$$\begin{tikzcd} F(\omega \times \omega) \arrow[loop left, "F(\tau)"] \arrow[r, shift left, "F\pi_1"] \arrow[r, shift right, "F\pi_2"'] & F\omega \arrow[l, bend right=60, "F\Delta"'] \end{tikzcd}$$

IS A GROUPOID IN $\mathcal{C}$.

GIVEN G, $[G, \mathcal{C}]$ IS THE CATEGORY OF "G-OBJECTS" IE PAIRS $(X \xrightarrow{f} G_0, G_1 \times_{G_0} X \xrightarrow{\alpha} X)$ WITH α AN ACTION (IE. UNIT AND ASSOCIATIVE LAWS).

GIVEN $f:\mathcal{F}\to\mathcal{E}$ DEFINE $DES_f(\mathcal{F})$ TO HAVE OBJECTS $(A\in \mathcal{O}b(\mathcal{F}), \theta: \pi_1^*A\to\pi_2^*A)$ WHERE $\pi_i:\mathcal{F}\times_{\mathcal{E}}\mathcal{F}\to\mathcal{F}$ AND θ SATISFIES UNIT & COCYCLE CONDITIONS. THERE IS A FUNCTOR $f^*:\mathcal{E}\longrightarrow DES_f(\mathcal{F})$ AND f IS AN EFFECTIVE DESCENT MORPHISM IF f^* IS AN EQUIVALENCE

2. THE JOYAL & TIERNEY REPRESENTATION THEOREM.

THEOREM [JT] IF $\mathcal{E}$ IS A GROTHENDIECK TOPOS OVER $\mathcal{S}$ THEN THERE EXISTS $\mathcal{G}$ A LOCALIC GROUPOID OVER $\mathcal{S}$ SUCH THAT

$$\mathcal{E}\simeq B\mathcal{G} \equiv \left\{ (X\xrightarrow{l} G_0,\ G_1\times_{G_0}X\xrightarrow{\alpha} X) \,\middle|\, l \text{ A LOCAL HOMEOMORPHISM } \alpha \text{ AN ACTION}\right.$$

PROOF OUTLINE

$$\begin{tikzcd}
\mathcal{F}\times_{\mathcal{E}}\mathcal{F} \arrow[drr, shift left, "\pi_1"] \arrow[drr, shift right, "\pi_2"'] \arrow[d, phantom, "\wr"] & & & & \\
Sh_{\mathcal{S}}(G_1) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & Sh_{\mathcal{S}}(G_0) \arrow[r, phantom, "\simeq"] & \mathcal{F} \arrow[r, "n", "\text{LOCALIC (PULLBACK OF LOCALIC)}"'] \arrow[d, "f"] & \mathcal{S}[\mathbb{N}\to\mathbb{O}] \arrow[d, "\text{EFFECTIVE DESCENT}"] \arrow[ddr, bend left, "\gamma", "\text{LOCALIC: POINTS OF } \mathbb{N}\to\mathbb{O} \text{ A POSET}"'] & \\
& & \mathcal{E} \arrow[r, "B", "\text{LOCALIC} \Leftrightarrow B \text{ A BOUND}"'] & \mathcal{S}[\mathbb{O}] \arrow[dr] & \\
& & & & \mathcal{S}
\end{tikzcd}$$

- f OF EFFECTIVE DESCENT (PULLBACK STABILITY)
- γn LOCALIC $\Rightarrow \exists$ LOCALE G_0 AS SHOWN
- f LOCALIC $\Rightarrow \pi_1$ LOCALIC $\Rightarrow \exists\, G_1$ AS SHOWN (& d_0, d_1 ETC)

SO $\mathcal{E} \simeq Des_f(\mathcal{F})$. OBJECTS OF $Des_f(\mathcal{F})$ ARE PAIRS $(A, \theta : \pi_1^* A \to \pi_2^* A)$ AND THESE CORRESPOND TO OBJECTS OF BG SINCE $\mathcal{F} \simeq Sh_{\mathcal{S}}(G_0)$ (AND OBJECTS OF $Sh_{\mathcal{S}}(G_0)$ ARE LOCAL HOMEOMORPHISMS OVER G_0). THE UNIT & COCYCLE CONDITIONS CORRESPOND TO 'α IS AN ACTION' IN THE DEFINITION OF BG.

$\square$

3. WHAT'S A 'LOCALIC' PROOF?

$$\begin{tikzcd}
\mathbf{TOP} \arrow[rr, "\simeq"] & & \mathbf{LOC} \\
\mathcal{F} \arrow[d, "f"'] \arrow[rr, mapsto] & & \underline{Loc}_{\mathcal{F}} \arrow[d, shift right=2, "\Sigma_f"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{E} \arrow[rr, mapsto] & & \underline{Loc}_{\mathcal{E}} \arrow[u, shift right=2, "f^*"']
\end{tikzcd}$$

$$\overbrace{\Sigma_f(W \times f^*X) \cong \Sigma_f W \times X}^{\text{FROBENIUS RECIPROCITY}}$$

$$f^* \$_{\mathcal{E}} \cong \$_{\mathcal{F}}$$

ALLOWS A REPRESENTATION OF GEOMETRIC MORPHISMS. IDEA IS TO REASON WITH THESE 'LOCALIC' REPRESENTATIONS.

EXAMPLES

$$\begin{tikzcd}
Sh_{\mathcal{S}}(X) \arrow[d, "f"'] \arrow[rr, mapsto] & & \underline{Loc}_{\mathcal{S}/X} \arrow[d, shift right=2, "\Sigma_f"'] \arrow[d, phantom, "\dashv"] \\
Sh_{\mathcal{S}}(Y) \arrow[rr, mapsto] & & \underline{Loc}_{\mathcal{S}/Y} \arrow[u, shift right=2, "f^*"']
\end{tikzcd}$$

– USUAL PULLBACK ADJUNCTION

$$\begin{tikzcd}
BG \arrow[d, "p"'] \arrow[rr, mapsto] & & {[G, \mathcal{S}]} \arrow[d, shift right=2, "\Sigma_p"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{S} \arrow[rr, mapsto] & & \underline{Loc}_{\mathcal{S}} \arrow[u, shift right=2, "p^*"']
\end{tikzcd}$$

$\Sigma_p(X,X)$ GIVEN BY COEQUALIZER

$$\begin{tikzcd}
G_1 \times_{G_0} X \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\pi_2"'] & X \arrow[r, two heads] & \Sigma_p(X,\alpha)
\end{tikzcd}$$

$(\underline{\text{NOTE}}: \Sigma_p(G_1, m) \cong G_0)$

$$\begin{tikzcd}
\mathcal{F} \arrow[d, "f"'] \arrow[rr, mapsto] & & \underline{Loc}_{\mathcal{F}} \arrow[d, shift right=2, "\Sigma_f"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{E} \arrow[rr, mapsto] & & \underline{Loc}_{\mathcal{E}} \arrow[u, shift right=2, "f^*"']
\end{tikzcd}$$

EFFECTIVE DESCENT – f $\iff$ f^* MONADIC.

[JT] ESSENTIALLY REDUCES TO:

LEMMA: $\mathcal{E}$ A GROTHENDIECK TOPOS OVER $\mathcal{S}$ ($\mathcal{E} \xrightarrow{p} \mathcal{S}$) THEN THERE EXISTS $\mathbb{G}$, LOCALIC GROUPOID OVER $\mathcal{S}$, SUCH THAT

$$\begin{tikzcd} \underline{\mathrm{Loc}}_{\mathcal{E}} \arrow[d, shift right=1.5ex, "\Sigma_p"'] \arrow[d, phantom, "\dashv"] & & {[\mathbb{G}, \mathcal{S}]} \arrow[d, shift right=1.5ex] \arrow[d, phantom, "\dashv"] \\ \underline{\mathrm{Loc}}_{\mathcal{S}} \arrow[u, shift right=1.5ex, "p"'] \arrow[rr, phantom, "\simeq"] & & \underline{\mathrm{Loc}}_{\mathcal{S}} \arrow[u, shift right=1.5ex] \end{tikzcd}$$

SAY "ESSENTIALLY" SINCE FROM $\underline{\mathrm{Loc}}_{\mathcal{E}} \simeq [\mathbb{G}, \mathcal{S}]$ ONE STILL NEEDS TO VERIFY THAT THE EQUIVALENCE RESTRICTS TO DISCRETE LOCALES / LOCAL HOMEOMORPHISMS. HOWEVER THIS FOLLOWS EASILY FROM THE CONSTRUCTION OF $\simeq$.

4 OUR "LOCALIC PROOF STRATEGY" STARTS WITH CHARACTERISIN THIS SITUATION:

LEMMA GIVEN $\mathcal{D} \underset{p^*}{\overset{\Sigma_p}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} \mathcal{C}$, $\mathcal{D}, \mathcal{C}$ CARTESIAN THEN $(\mathcal{D} \rightleftarrows \mathcal{C})$ IS EQUIVALENT TO $[\mathbb{G}, \mathcal{C}] \underset{p^*}{\overset{\Sigma_p}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} \mathcal{C}$ FOR SOME GROUPOID $\mathbb{G}$ IN $\mathcal{C}$ IFF $\exists W \twoheadrightarrow 1$ OF EFFECTIVE DESCENT (IE $W^*: \mathcal{D} \to \mathcal{D}/W$ MONADIC) SUCH THAT $\mathcal{D}/W \xrightarrow{\Sigma_{\bar{p}}} \mathcal{C}/\Sigma_p W$ IS AN EQUIVALENCE OF CATEGORIES, WHERE $\Sigma_{\bar{p}}(W' \xrightarrow{f} W) \equiv \Sigma_p W' \xrightarrow{\Sigma_p f} \Sigma_p W$.

PROOF ⇒ (I.E. GIVEN $[\mathcal{G},\mathcal{C}] \underset{p^*}{\overset{\Sigma_p}{\rightleftarrows}} \mathcal{C}$) $\perp$

CONSIDER THE OBJECT (G_1, m) OF $[\mathcal{G},\mathcal{C}]$

RECALL:
$\Sigma_p(G_1,m) \cong G_0$

$$\begin{tikzcd}
{[\mathcal{G},\mathcal{C}]/(G_1,m)} \arrow[r, shift left, "\Sigma_{(G_1,m)}"] \arrow[r, phantom, "\perp"] \arrow[d, "\Sigma_{\bar{p}}"'] & {[\mathcal{G},\mathcal{C}]} \arrow[l, shift left, "(G_1,m)^*"] \\
(\mathcal{C}/G_0) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & (\mathcal{C}/G_0)^{\mathbb{T}} \arrow[l, shift left]
\end{tikzcd}$$

12 ⇒ $\underline{(G_1,m)^*}$ $\underline{\text{MONADIC}}$

$\mathbb{T}$ – INDUCED BY $\mathcal{G}$

PROVE THIS IS AN EQUIVALENCE JUST AS YOU WOULD PROVE, FOR ANY GROUP $(G,*)$:

$$\boxed{\begin{aligned}
G\text{-}\underline{Set}/(G,*) &\xrightarrow{\;\simeq\;} \underline{Set} \\
[(X,\alpha) \xrightarrow{n} (G,*)] &\longmapsto \mathrm{Ker}(n) \\
[(G\times X, *\times \mathrm{Id}) \xrightarrow{\pi_1} (G,*)] &\longmapsfrom X
\end{aligned}}$$

PROOF CONT. $\Leftarrow$

$$\begin{tikzcd}
W \times W \arrow[loop left, "\tau"] \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & W & \mathcal{D}/W \arrow[r, shift left, "\Sigma_W"] \arrow[r, phantom, "\perp"] \arrow[d, "\Sigma_{\bar{p}}"] & \mathcal{D} \arrow[l, shift left, "W^*"] \\
& \Downarrow & \mathcal{E}/\Sigma_p W &
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma_p(W \times W) \arrow[loop left, "\Sigma_p(\tau)"] \arrow[r, shift left, "\Sigma_p \pi_1"] \arrow[r, shift right, "\Sigma_p \pi_2"'] \arrow[d, phantom, "\cong"] & \Sigma_p W \arrow[d, phantom, "\cong"] \\
G_1 \arrow[loop left, "i"] \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & G_0
\end{tikzcd}$$

A GROUPOID AS PER ~~BE~~ EARLIER EXAMPLE.

$\bar{\pi}'$ THE 'IMAGE' OF $\bar{\pi}$ VIA THE EQUIVALENCE $\Sigma_{\bar{p}}$

$$\mathcal{D} \simeq (\mathcal{D}/W)^{\bar{\pi}} \xrightarrow[\simeq]{\text{INDUCED BY } \Sigma_W \dashv W^*} (\mathcal{E}/G_0)^{\bar{\pi}'} = [\mathbb{G}, \mathcal{E}] \qquad \square$$

THIS TECHNIQUE IS KNOWN; [SGA 4]: $\mathcal{E}$ IS ÉTENDU IF $\exists\, B \twoheadrightarrow 1$ $\mathcal{E}$ S.T. $\mathcal{E}/B \simeq Sh_S(G_0)$. SO IF $\mathcal{E}$ IS ÉTENDU THEN:

$$\begin{tikzcd}
\mathcal{E}/B \times B \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, phantom, "\cong"] & \mathcal{E}/B \arrow[r] \arrow[d, phantom, "\cong"] & \mathcal{E} \\
Sh(G_1) \arrow[r, shift left] \arrow[r, shift right] & Sh(G_0) &
\end{tikzcd}$$

$\Rightarrow \mathcal{E} \simeq B\mathbb{G}$.

SO BY THE LEMMA [JT] REDUCES TO: $\mathcal{E}$ GROTHENDIECK $\Rightarrow \exists W \rightarrow$ OF EFFECTIVE DESCENT IN $\underline{Loc}_{\mathcal{E}}$ S.T. $\underline{Loc}_{\mathcal{E}/W} \xrightarrow{\Sigma_{\bar{p}}} \underline{Loc}_{\mathcal{S}/\Sigma_p W}$ IS AN EQUIVALENCE.

NOTE THAT BY OUR PROOF GIVEN ABOVE OF [JT] WE KNOW WHAT W 'SHOULD BE' IE TAKE $W = [\mathbb{N} \twoheadrightarrow B]$, THE LOCALE OF SURJECTIONS FROM $\mathbb{N}$ TO B. I.E. THIS SHOWS US HOW TO GET W (AND SO G) FROM THE BOUND B.

TO REDUCE FURTHER, WHAT IS THE LOCALIC INTERPETATION OF $\underline{Loc}_{\mathcal{E}/W} \xrightarrow{\Sigma_{\bar{p}}} \underline{Loc}_{\mathcal{S}/\Sigma_p W}$? WELL THERE IS AN ADJUNCTIO

$$\begin{tikzcd} \underline{Loc}_{\mathcal{E}/W} \arrow[r, shift left=2, "\Sigma_{\bar{p}}"] \arrow[r, phantom, "\perp"] & \underline{Loc}_{\mathcal{S}/\Sigma_p W} \arrow[l, shift left=2, "{\eta_W^*[p^*(\_)]}"] \end{tikzcd} \qquad -①$$

IE RIGHT ADJOINT GIVENBY –

$$\begin{tikzcd} X \arrow[d, "n"'] & & \eta_W^* p^*(n) \arrow[r] \arrow[d] & p^* X \arrow[d, "p^*(n)"] \\ \Sigma_p W & \longmapsto & W \arrow[r, "\eta_W"] & p^* \Sigma_p W \end{tikzcd}$$

WHERE η IS UNIT OF $\Sigma_p \dashv p^*$. NOTE THAT UNIT OF (1), AT $f: Y \rightarrow W$ IS

$$Y \xrightarrow{(f, \eta_Y)} W \times_{p^* \Sigma_p W} p^* \Sigma_p Y$$

SO, CONCERNING $\Sigma_{\bar{p}}$, WE KNOW

① $\Sigma_{\bar{p}}$ IS PART OF AN ADJUNCTION: $\Sigma_{\bar{p}} \dashv \bar{p}^*$

② $\Sigma_{\bar{p}}(1) = 1$

③ $\Sigma_{\bar{p}} \dashv \bar{p}^*$ SATISFIES FROBENIUS RECIPROCITY – IT IS REPRESENTATION OF $\bar{p} : \mathrm{Sh}_{\mathcal{E}}(W) \to \mathrm{Sh}_{\mathcal{E}}(\Sigma_p$

BUT CONSIDER THE LEMMA:

LEMMA: $L \dashv R : \mathcal{D} \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} \mathcal{C}$ AN ADJUNCTION BETWEEN CARTESIAN CATEGORIES, WITH $L1 = 1$ AND $L \dashv R$ SATISFYING FROBENIUS RECIPROCITY, THEN η REGULAR MONIC $\Rightarrow$ L IS AN EQUIV.

PROOF: $LRX \cong L(1 \times RX) \cong L1 \times X \cong X \Rightarrow \varepsilon$ ISO.

SO BY TRIANGULAR IDENTITIES $L\eta$ IS ISO. BUT THERE IS AN EQUALIZER $W \xrightarrow{\eta_W} RLW \underset{a_2}{\overset{a_1}{\rightrightarrows}} W' \Rightarrow La_1 = La_1$ SINCE $L\eta$ ISO. BUT THEN $a_1 = a_2$ AS $\eta_{W'}$ MONO.

$\Rightarrow \eta_W$ ISO $\Rightarrow L$ AN EQUIVALENCE □

AND SO [JT] REDUCES TO THE FOLLOWING LEMMA:

LEMMA: $\mathcal{E}$ A GROTHENDIECK TOPOS OVER $\mathcal{S}$ (VIA $p : \mathcal{E} \to \mathcal{S}$) THEN $\exists W \twoheadrightarrow 1$ OF EFFECTIVE DESCENT IN $\underline{\mathrm{Loc}}_{\mathcal{E}}$ SUCH THAT $\forall f : Y \to W$ $Y \xrightarrow{(f, \eta_Y)} W \times_{p^*\Sigma_p W} p^*\Sigma_p Y$.

IN PRACTICE WE WOULD GUESS TO TAKE $W = [\mathbb{N} \to B]$.

SO <u>IN SUMMARY</u> AN ENTIRELY LOCALIC ACCOUNT OF [JT] CAN BE GIVEN IF WE CAN SHOW THE FOLLOWING TWO CONDITIONS ARE EQUIVALENT FOR ANY OBJECT B OF $\mathcal{E}$ AND GEOMETRIC MORPHISM $p: \mathcal{X} \longrightarrow \mathcal{S}$.

Ⓘ

$\forall f: Y \longrightarrow [\mathbb{N} \to B]$

$$\Omega[\mathbb{N} \to B] \otimes_{\Omega p^* \Sigma_p [\mathbb{N} \to B]} \Omega p^* \Sigma_p Y \xrightarrow{\Omega(f, \xi_Y)} \Omega Y$$

A SURJECTION

(II)

$\forall A \in Ob(\mathcal{E})$

$$B \times p^* p_* (\tilde{A}^B) \xrightarrow{Id \times \varepsilon^{gm}} B \times \tilde{A}^B \xrightarrow{ev} \tilde{A}$$

A SURJECTION

FOR Ⓘ RECALL THAT A LOCALE MAP $f: X \to Y$ IS A REG. MONIC IFF $\Omega f: \Omega Y \to \Omega X$ IS A SURJ.

FOR (II) CONSULT B3.1.6 OF JOHNSTONE'S 'ELEPHANT' FOR THE EASY PROOF THAT THIS IS EQUIVALENT TO THE CONDITIO 'B IS A BOUND FOR p'. $\tilde{A}$ IS THE PARTIAL MAP CLASSIFIER ON A AND ε^{gm} IS THE COUNIT OF $p^* \dashv p_*$. ε^{gm} FACTORS VIA $\Omega\mathcal{X}$, BUT THE EQUIVALENCE OF Ⓘ AND (II) HAS NOT BEEN CONFIRMED.

11-SEP-11