EFFECTIVE DESCENT VIA NATURAL TRANSFORMATIONS

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OBJECTIVE:

OUTLINE AN AXIOMATIC PROOF THAT TRIQUOTIENT SURJECTIONS ARE OF EFFECTIVE DESCENT

CONTEXT / NOTATION

$$\mathbb{C} \sim \mathrm{Loc}$$

WHERE $\mathrm{Loc} \equiv (\mathrm{Frm})^{op}$ (FRAMES)

FRAME $\equiv$ COMPLETE HEYTING ALGEBRA

FRAME HOM. $\equiv$ PRESERVES $\bigvee$ AND $\wedge$.

<u>WHY Loc?</u> E.G. STONE-ČECH COMPACTIFICATION.

<u>AXIOMATICALLY</u>

① $\mathbb{C}$ HAS FINITE LIMITS / COLIMITS

② $\exists\ \$ \in \mathrm{DLat}(\mathbb{C})$.

<u>NOTATION</u>:

$$\mathbb{C} \xrightarrow{\$^{(-)}} [\mathbb{C}^{op}, \mathrm{Set}]$$

$$X \longmapsto (Y \longmapsto \mathbb{C}(Y \times X, \$))$$

<u>NOTE</u>: $\$^X$ IS $\mathbb{C}(-,\$)^{\mathbb{C}(-,X)}$ IN $[\mathbb{C}^{op}, \mathrm{Set}]$

AXIOMS TRUE OF LOC?

LOC HAS FINITE LIMITS / COLIMITS ✓

$\exists \$ \in \mathrm{DLAT}(\mathrm{Loc})$; TAKE $\$$ TO BE THE SIERPIŃSKI LOCALE, I.E.

$$\Omega\$ \equiv \{0 \leq * \leq 1\}$$

IMPORTANCE OF $\$^{(-)} : \mathbb{C} \longrightarrow [\mathbb{C}^{op}, \mathrm{Set}]$ FOR $\mathbb{C} = \mathrm{Loc}$ IS:

$\forall$ LOCALES X, Y

$$\mathrm{NAT}\left[\$^X, \$^Y\right] \cong \mathrm{DCPO}(\Omega X, \Omega Y)$$

WHERE DCPO HOM. $\equiv$ $\hat{\bigvee}$ PRESERVING.

(VICKERS/TOWNSEND)

IF $\mathbb{C}$ = LOC $\quad \mathrm{NAT}[\$^X, \$^Y] \cong \mathrm{DCPO}(\Omega X, \Omega Y)$.

APPLICATION:

DEF$^{\underline{N}}$ (PLEWE) $f : X \to Y$ IN LOC A TRIQUOTIENT SURJECTION IFF $\exists\, f_\# : \Omega X \to \Omega Y$ DCPO HOM. WITH

1. $f_\#(a_1 \wedge (a_2 \vee \Omega f(b))) = [f_\# a_1 \wedge b] \vee f_\#(a_1 \vee a_2)$
2. $f_\# \Omega f = \mathrm{Id}$

$$\Big\downarrow$$

VIA NATURAL TRANSFORMATIONS

DEF$^{\underline{N}}$ $f : X \to Y$ IN $\mathbb{C}$ A TRIQUOTIENT SURJECTION IFF $\exists\, f_\# : \$^X \longrightarrow \$^Y$ WITH

1. $f_\#(a_1 \wedge (a_2 \vee \$^f(b))) = [f_\# a_1 \wedge b] \vee f(a_1 \vee a_2)$
2. $f_\# \$^f = \mathrm{Id}$

IN $[\mathbb{C}^{op}, \mathrm{SET}]$

Effective Descent

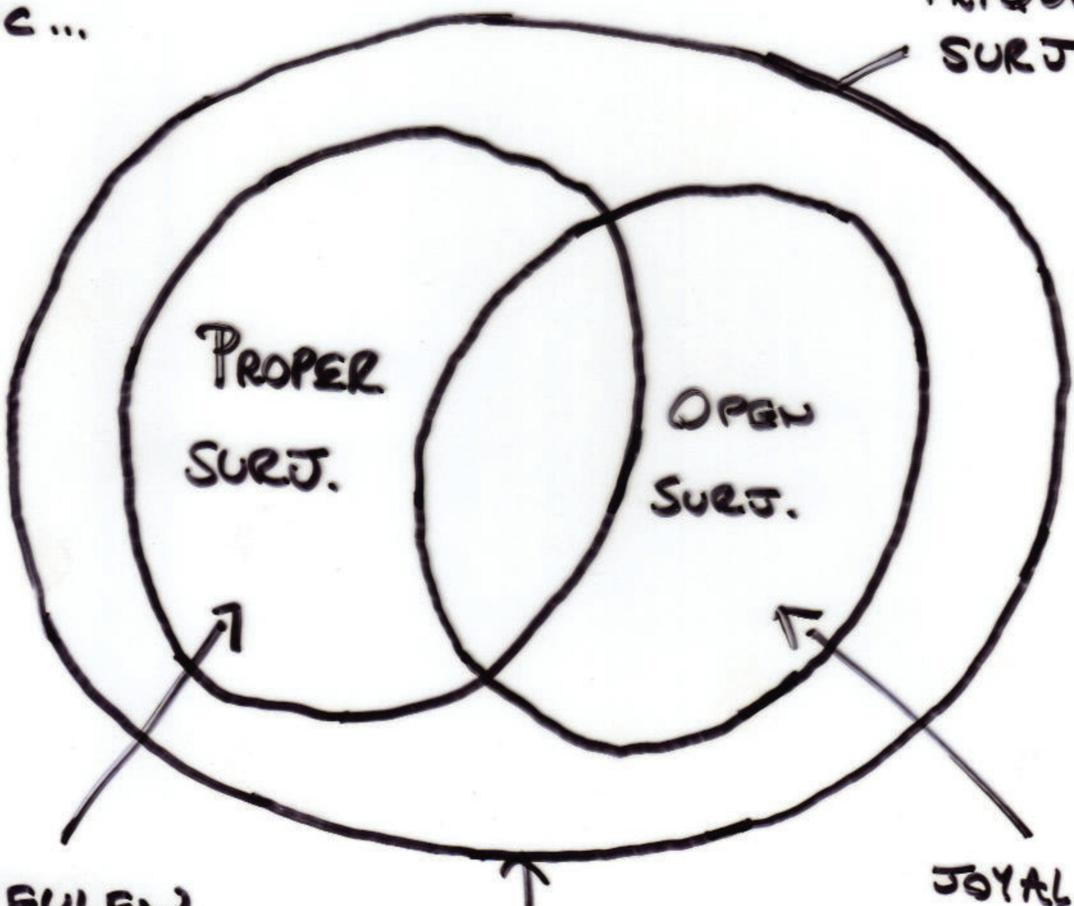
Given $f: X \to Y$

$$\begin{tikzcd}
\mathcal{C}/X \arrow[rr, shift left, "\Sigma_f"] \arrow[rr, phantom, "\perp"] \arrow[dr, shift left] & & \mathcal{C}/Y \arrow[ll, shift left, "f^*"] \arrow[dl, "\simeq"] \\
& (\mathcal{C}/X)^{\mathbb{T}} \arrow[ul, shift left] &
\end{tikzcd}$$

... Interesting categorical question.
... Useful class of pullback stable quotients (topology)

All of Effective Descent in Loc...

Triquotient surjections.



Vermeulen "Proper Maps" JPAA

Plewe '97 Camb. Proc.

Joyal & Tierney 1984

WHEN EFFECTIVE DESCENT?

CAN USE BECK'S

MONADICITY CRITERIA

$$\begin{tikzcd} & \mathcal{C}/Y \arrow[dl, shift left, "f^*"] \arrow[d, "K"] \\ \mathcal{C}/X \arrow[ur, shift left, "\Sigma_f"] \arrow[r, shift left] & (\mathcal{C}/X)^T \arrow[l, shift left] \end{tikzcd}$$

THAT IS, ① & ② WHERE

① $f^* : \mathcal{C}/Y \longrightarrow \mathcal{C}/X$ REFLECTS ISOMORPHISMS

$\uparrow$ E.G. A1.3.2 ELEPHANT, JOHNSTONE

$\boxed{f \text{ PULLBACK STABLE REGULAR EPIMORPHISM.}}$

② GIVEN $A \underset{a_2}{\overset{a_1}{\rightrightarrows}} B \xrightarrow{c} C$ COEQUALIZER IN $\mathcal{C}/Y$

WITH THE COEQUALIZER

$$f^*A \underset{f^*a_2}{\overset{f^*a_1}{\rightrightarrows}} f^*B \xrightarrow{q} Q$$

<u>SPLIT</u> IN $\mathcal{C}/X$ THEN

$$A \underset{a_2}{\overset{a_1}{\rightrightarrows}} B \xrightarrow{c} C$$ IS PULLBACK STABLE COEQUALIZER.

POSSIBLE THEOREM ABOUT $\mathbb{C}$

$f : X \to Y$ TRIQUOTIENT SURJECTION WITH $f_{\#} : \$^X \to \$^Y$ THEN FOR ANY PULLBACK SQ.

$$\begin{tikzcd} Z \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

THERE EXISTS UNIQUE $(p_1)_{\#} : \$^{Z \times_Y X} \longrightarrow \$^Z$

MAKING p_1 TRIQUOTIENT SURJECTION AND BECK - CHEVALLEY; I.E.

$$(p_1)_{\#} \$^{p_2} = \$^g f_{\#}$$

/

<u>TRUE</u> FOR $\mathbb{C} = \mathrm{Loc}$ (PLEWE + VICKERS GENERALIZED)

ALSO TRUE WITH SUITABLE AXIOMS ON $\mathbb{C}$.

PSSL '79. NEED TO FORCE \$ TO BEHAVE LIKE TOPOLOGICAL \$.

QUESTION WHAT EXTRA AXIOMS DO WE NEED TO ENSURE TRIQUOTIENT SURJECTIONS ARE REGULAR EPIMORPHISMS?

AXIOM 1 GIVEN $A \overset{q_1}{\underset{q_2}{\rightrightarrows}} B \xrightarrow{c} C$ COEQUALIZER IN $\mathbb{C}$

$$\$^C \xhookrightarrow{\$^c} \$^B \overset{\$^{q_1}}{\underset{\$^{q_2}}{\rightrightarrows}} \$^A$$

IS AN EQUALIZER IN ...

$$\mathbb{C}^{op}_{IP} \subseteq [\mathbb{C}^{op}, \mathrm{SET}]$$

FULL SUBCATEGORY OF OBJECTS $\$^X$.

TRUE IN LOC? YES ...

$$\begin{tikzcd}
\Omega C \arrow[r, hook, "\Omega c"] & \Omega B \arrow[r, shift left, "\Omega q_1"] \arrow[r, shift right, "\Omega q_2"'] & \Omega C \\
\Omega E \arrow[u, dotted, "\exists!"] \arrow[ur, "DCPO"'] & &
\end{tikzcd}$$

EQUALIZER IN FRAME

- CREATED IN SET.

__LEMMA__ $f : X \to Y$ TRIQUOTIENT SURJECTION THEN f REGULAR EPIMORPHISM.

__PROOF__

$$\begin{tikzcd}
X \times_Y X \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X \arrow[r, "q"] \arrow[dr, "f"'] & D \arrow[d, dotted] \\
& & Y
\end{tikzcd}$$

CERTAINLY THERE EXISTS $\$^i : \$^Y \to \$^D$ AND

$$\$^D \xrightarrow{\$^q} \$^X \xrightarrow{f_*} \$^Y .$$

BUT THESE ARE INVERSE SINCE

$$(f_* \$^q)(\$^i) = f_* \$^f = Id$$

DEF$^{\underline{N}}$ TRIQUOTIENT.

__AND__ $\$^q \, \$^i f_* \$^q = \$^q$

BECK-CHEVALLEY

$\Rightarrow (\$^i)(f_* \$^q) = Id$ SINCE $\$^q$ MONOMORPHISM BY AXIOM 1.

HENCE $\$^D \cong \$^Y \overset{???}{\Rightarrow} D \cong Y$

NEED FURTHER AXIOM.

AXIOM 2. $\$^{(-)} : \mathbb{C} \longrightarrow [\mathbb{C}^{op}, \mathbf{Set}]$ REFLECTS ISOMORPHISMS.

TRUE OF LOC? YES...

$$\Omega X \cong_{DCPO} \Omega Y \Rightarrow \Omega X \cong_{Pos} \Omega Y$$

$$\Rightarrow X \cong Y \text{ IN LOC.}$$

HENCE GIVEN EXTRA AXIOMS (1) & (2):-

$$\boxed{\text{TRIQUOTIENT SURJ.} \Rightarrow \text{PULLBACK STABLE REGULAR EPIMORPHISM.}}$$

$$\Downarrow$$

ONE HALF OF BECK'S MONADICITY CRITERIA SATISFIED; I.E. $f^* : \mathbb{C}/Y \longrightarrow \mathbb{C}/X$ CONSERVATIVE.

FINALLY FOR 2^{ND} HALF BECK'S CRITERIA

SAY
$$\begin{tikzcd} A \arrow[r, shift left, "a_1"] \arrow[r, shift right, "a_2"'] & B \arrow[r, "c"] & C \end{tikzcd}$$
COEQUALIZER IN $\mathbb{C}/Y$

WITH
$$\begin{tikzcd} A\times_Y X \arrow[r, shift left, "a_1\times 1"] \arrow[r, shift right, "a_2\times 1"'] & B\times_Y X \arrow[r, "q"] \arrow[l, bend left=50] & Q \arrow[l, bend left=50] \end{tikzcd}$$
SPLIT COEQUALIZER

THEN BY AXIOM 1
$$\begin{tikzcd} \$^C \arrow[r, hook, "\$^c"] & \$^B \arrow[r, shift left, "\$^{a_1}"] \arrow[r, shift right, "\$^{a_2}"'] & \$^A \end{tikzcd}$$
IS AN EQUALIZER (IN FULL SUBCATEGORY OF $[\mathbb{C}^{op}, Set]$)

DETAILS OMITTED

WE CAN DRAW...
$$\begin{tikzcd} \$^C \arrow[r, hook, "\$^c"] & \$^B \arrow[r, shift left, "\$^{a_1}"] \arrow[r, shift right, "\$^{a_2}"'] & \$^A \\ \$^Q \arrow[u, dotted, "x"] \arrow[r, hook, "\$^q"] & \$^{B\times_Y X} \arrow[u, "(\pi_1^B)_\#"] \arrow[r, shift left, "\$^{a_1\times 1}"] \arrow[r, shift right, "\$^{a_2\times 1}"'] & \$^{A\times_Y X} \arrow[u, "(\pi_1^A)_\#"'] \end{tikzcd}$$

THEREFORE THE MAP:
$$\$^B \xrightarrow{\$^{\pi_1^B}} \$^{B\times_Y X} \xrightarrow{q_\#} \$^Q \xrightarrow{x} \$^C$$

ENSURES $\$^c$ COSPLIT REG. MONOMORPHISM IN $[\mathbb{C}^{op}, Set]$.

EXISTS SINCE q SPLIT.

SUMMARY

- BY ADDING AXIOMS (1) & (2), ON COEQUALIZERS AND ON $\$^{(-)}$ BEING CONSERVATIVE, WE HAVE AXIOMATIC PROOF OF THE FACT (PLEWE) THAT TRIQUOTIENT SURJECTIONS ARE EFFECTIVE DESCENT MORPHISMS.

- COVERS PROPER AND OPEN SURJECTIONS

- <u>NOT</u> AS GENERAL AS MOERDIJKS [BULL SOC. MATH BELGIQUE '89]

- FINITE LOCALES ALSO MODEL AXIOMS.

  $\Rightarrow$ NOT EVERY EFFECTIVE DESCENT MORPHISM IS TRIQUOTIENT SURJECTION.

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