A Categorical Account of the Localic Closed Subgroup Theorem

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MOTIVATION

$Loc \equiv (Frames)^{op}$
frame $=$ cHa, ΩX

$\mathbb{Q} \subseteq \mathbb{R}$, $\mathbb{Q}$ NOT CLOSED

IN CONTRAST TO TOPOLOGY, IN LOCALE THEORY, EVERY LOCALIC SUBGROUP IS CLOSED. [ISBELL, KŘÍŽ, PULTR, ROSICKÝ]

SO, WHILST LOCALE THEORY LOOKS LIKE TOPOLOGY, IT IS FUNDAMENTALLY DIFFERENT

STONE-ČECH, COMPACT HAUSDORFF, VIETORIS

RELATIONSHIP BETWEEN ALGEBRAIC STRUCTURE & PURELY TOPOLOGICAL

QUESTION

- CATEGORICAL APPROACHES EXIST TO LOCALE THEORY, RESULT STILL HOLD?

$\mathcal{C}$ ORDER ENRICHED
$\exists \$ \in Ob(\mathcal{C}),\ P_L X \hookrightarrow \$^{\$^X}$

- THESE APPROACHES ARE ORDER DUAL; WHAT IS DUAL?

ALSO FOUND

- HOFMANN - MISLOVE THEOREM KEY STEP
- NON-STANDARD DEFINITION OF DENSE.

Locale Theory

Loc order enriched, finite limits

$\$$, Sierpiński locale, is order internal DLat.

$\Delta \dashv \Pi,\ \sqcup \dashv \Delta$

$\$$ classifies closed & open subobjects.

Vickers/Townsend For any frames ΩX & ΩY,

$$\mathrm{SUP}(\Omega X, \Omega Y) \cong \sqcup\text{-SLAT}[\$^X, \$^Y] \text{ ---- in } [\mathrm{Loc}^{op}, \mathrm{Set}].$$

$\bigvee$ preserving

$$\mathrm{PreFr}(\Omega X, \Omega Y) \cong \sqcap\text{-SLAT}[\$^X, \$^Y]$$

$\hat{\bigvee}$ & $\wedge$ preserving

$\$^X$ is $(Y\$)^{YX}$.

Idea. Any definition/fact in locale theory that involves interaction between sup-lattice homomorphisms and locale maps can be replicated in any $\mathcal{C}$ with $\$ \in \mathrm{DLat}(\mathcal{C})$.

Aside

$\$^X \cong \$^Y \Rightarrow X \cong Y$

$X_0 \overset{i}{\hookrightarrow} X \Rightarrow \$^X \overset{\$^i}{\twoheadrightarrow} \$^{X_0}$

$[\mathcal{C}^{op}, \mathrm{Set}]$

$\mathcal{C}$

Facts About Loc.

$\exists\, P_L$, LOWER POWER MONAD.

IT IS KZ.

$Loc(Y, P_L X) \cong SUP(\Omega X, \Omega Y)$

$\Omega(X \times Y) \cong \Omega X \otimes_{SUP} \Omega Y$

Axioms on $\mathcal{C}$

$\exists\, P_L$, A KZ MONAD ON $\mathcal{C}$ SUCH THAT,

$\mathcal{C}(Y, P_L X) \cong U\text{-SLAT}[\$^X, \$^Y]$

$$\$^X \times \$^Y \xrightarrow{\$^{\pi_1} \times \$^{\pi_2}} \$^{X\times Y} \times \$^{X \times Y} \xrightarrow{\sqcap} \$^{X \times Y}$$

IS UNIVERSAL U-BILINEAR

Definitions

$f: X \to Y$ OPEN IF THERE EXISTS $\exists_f : \Omega X \to \Omega Y$, $\exists_f \dashv \Omega f$ AND

$\exists_f(a \wedge \Omega f\, b) = \exists_f a \wedge b$

FOR ALL $a \in \Omega X$, $b \in \Omega Y$

$f: X \to Y$ OPEN IF $\exists_f : \$^X \to \$^Y$, $\exists_f \dashv \$f$ WITH

$$\begin{tikzcd}
\$^X \times \$^Y \arrow[rr, "\exists_f \times Id"] \arrow[d, "Id \times \$f"'] & & \$^Y \times \$^Y \arrow[d, "\sqcap"] \\
\$^X \times \$^X \arrow[r, "\sqcap"] & \$^X \arrow[r, "\exists_f"] & \$^Y
\end{tikzcd}$$

X OPEN IF $!: X \to 1$ OPEN ($\exists_X$ NOT $\exists_!$)

X OPEN IF $!: X \to 1$ OPEN.

$X_0 \hookrightarrow X$ FITTED IF

$$\begin{tikzcd}
X_0 \arrow[r, hook] & X \arrow[rr, "f"] \arrow[dr, "!"'] & \arrow[d, phantom, "\sqsupseteq"] & Y \\
& & 1 \arrow[ur, "p"'] &
\end{tikzcd}$$

$X_0 \hookrightarrow X$ FITTED IF

$$\begin{tikzcd}
X_0 \arrow[r, hook] & X \arrow[rr, "f"] \arrow[dr, "!"'] & \arrow[d, phantom, "\sqsupseteq"] & Y \\
& & 1 \arrow[ur, "p"'] &
\end{tikzcd}$$

Theorems

- OPEN SUBOBJECTS FITTED
- OPEN MAP PULLBACK STABLE

FACTS ABOUT LOC

$\exists\ P_U$, UPPER POWER MONAD.

IT IS coKZ.

$Loc(Y, P_L X) \cong$ [redacted] Preframe $(\Omega X, \Omega Y)$

$\Omega(X \times Y) \cong \Omega X \otimes_{Prefr} \Omega Y$

AXIOMS ON $\mathcal{C}$

P_U, A coKZ MONAD SUCH THAT

$\mathcal{C}(Y, P_L X) \cong \sqcap\text{-SLAT}[\$^X, \$^Y]$

$\$^X \times \$^Y \xrightarrow{\$^{\pi_1} \times \$^{\pi_2}} \$^{X\times Y} \times \$^{X \times Y} \xrightarrow{\sqcup} \$^{X \times Y}$

IS UNIVERSAL [redacted] $\sqcap$-BILINEAR.

DEFINITIONS

$f : X \to Y$ PROPER IF THERE EXISTS $\forall_f : \Omega X \to \Omega Y$ PREFRAME HOM. $\Omega f \dashv \forall_f$ & $\forall_f(a \vee \Omega f\, b) = \forall_f a \vee b$ FOR ALL $a \in \Omega X$, $b \in \Omega Y$.

$f : X \to Y$ PROPER IF $\exists$ $\forall_f : \$^X \to \$^Y$, $\$f \dashv \forall_f$ &

$$\begin{tikzcd}
\$^X \times \$^Y \arrow[rr, "\forall_f \times \mathrm{Id}"] \arrow[d, "\mathrm{Id} \times \$f"'] & & \$^Y \times \$^Y \arrow[d, "\sqcup"] \\
\$^X \times \$^X \arrow[r, "\sqcup"] & \$^X \arrow[r, "\forall_f"] & \$^Y
\end{tikzcd}$$

[redacted]

X COMPACT IF $! : X \to 1$ PROPER.

X COMPACT IF $! : X \to 1$ PROPER

$X_0 \hookrightarrow X$ WEAKLY CLOSED IF

$$\begin{tikzcd}
X_0 \arrow[r, hook] & X \arrow[rr, "f"] \arrow[dr, "!"'] & \sqcap\!\mathrm{I} & Y \\
& & 1 \arrow[ur, "p"'] &
\end{tikzcd}$$

$X_0 \hookrightarrow X$ WEAKLY CLOSED IF

$$\begin{tikzcd}
X_0 \arrow[r, hook] & X \arrow[rr, "f"] \arrow[dr, "!"'] & \sqcap\!\mathrm{I} & Y \\
& & 1 \arrow[ur, "p"'] &
\end{tikzcd}$$

THEOREMS

- CLOSED SUBOBJECTS WEAKLY CLOSED
- PROPER MAPS ARE PULLBACK STABLE.

Theorem. Axiomatically on $\mathcal{C}$

$$\mathcal{C}(1, P_U X) \cong \{ X_0 \overset{i}{\hookrightarrow} X \mid X_0 \text{ compact}, i \text{ fitted} \}$$

$$\mathcal{C}(1, P_L X) \cong \{ X_0 \hookrightarrow X \mid X_0 \text{ open}, i \text{ weakly closed} \}$$

(Hofmann–Mislove).

Application: Any $X_0 \overset{i}{\hookrightarrow} X$ in $\mathcal{C}$ with X_0 open factors uniquely as

$$X_0 \overset{j}{\hookrightarrow} \overline{X}_0 \overset{\bar{i}}{\hookrightarrow} X$$

j dense — $X_0, \overline{X}_0$ open &

$\bar{i}$ weakly closed

$$\begin{tikzcd}
\$^{X_0} \arrow[dr, "\exists_{X_0}"'] & & \$^X \arrow[ll, "\$^j"'] \arrow[dl, "\exists_X"] \\
& \$ &
\end{tikzcd}$$

Proof

X_0 open $\Rightarrow$ $\$^X \xrightarrow{\$^i} \$^{X_0} \xrightarrow{\exists_{X_0}} \$$ $\cup$-SLat Hom.

$\Rightarrow$ there exists $p : 1 \longrightarrow P_L X$

Construct

$$\begin{tikzcd}
\overline{X}_0 \arrow[r, hook, "\bar{i}"] & X \arrow[rr, "\eta^L_X"] \arrow[dr, "!"'] & \arrow[d, phantom, "\equiv"] & P_L X \\
& & 1 \arrow[ur, "p"'] &
\end{tikzcd}$$

//

$\underline{Loc}_{Sh(Z)} \simeq \underline{Loc}/Z$, SIMILARLY $\mathcal{C}$.

RELATIVE DENSENESS.

GIVEN
$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "fj"'] & & Y \arrow[dl, "f"] \\ & Z & \end{tikzcd}$$
IN $\mathcal{C}$, j IS DENSE OVER f IF f OPEN & fj OPEN &
$$\begin{tikzcd} \$^X \arrow[dr, "\exists_{fj}"'] & & \$^Y \arrow[ll, "\$j"'] \arrow[dl, "\exists_f"] \\ & \$^Z & \end{tikzcd}$$
COMMUTES.

$Z=1$

LEMMA IF j IS DENSE OVER f AND Z IS OPEN $\Rightarrow$ j IS DENSE.

PROOF
$$\begin{tikzcd} \$^X \arrow[dr, "\exists_{fj}"] \arrow[ddr, bend right, "\exists_X"'] & & \$^Y \arrow[ll, "\$j"'] \arrow[dl, "\exists_f"'] \arrow[ddl, bend left, "\exists_Y"] \\ & \$^Z \arrow[d, "\exists_Z"] & \\ & \$ & \end{tikzcd}$$
//

LEMMA GIVEN PULLBACKS
$$\begin{tikzcd} X' \arrow[r, "j'"] \arrow[d, "c"] & Y' \arrow[r, "f'"] \arrow[d, "b"] & Z' \arrow[d, "a"] \\ X \arrow[r, "j"] & Y \arrow[r, "f"] & Z \end{tikzcd}$$
IF j DENSE OVER f THEN j' DENSE OVER f'.

FURTHER: THE INTERSECTION OF TWO RELATIVELY DENSE SUBOBJECTS IS RELATIVELY DENSE.

PROOF: BY PULLBACK STABILITY OF OPEN MAPS & BECK-CHEVALLEY. //

Closed Subgroup Theorem for $\mathcal{C}$.

Theorem: $H \overset{i}{\hookrightarrow} G$ a subgroup and H open then i weakly closed.

Proof Factor i as $H \overset{j}{\hookrightarrow} \bar{H} \overset{\bar{i}}{\hookrightarrow} G$ ...... $\boxed{HM \Rightarrow \bar{H} \text{ open.}}$
then show j isomorphism.

Key is to consider pullback squares

$$\begin{tikzcd}
H \times H \arrow[r, hook, "j \times Id"] \arrow[d, "\pi_1"'] & \bar{H} \times \bar{H} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \bar{H} \arrow[d, "!"] \\
H \arrow[r, hook, "j"] & \bar{H} \arrow[r, "!"] & 1
\end{tikzcd}$$

Then $j \times Id$ dense over $\pi_2 \Rightarrow j \times Id$ dense
symmetrically $Id \times j$ dense $\Rightarrow j \times j$ dense
So consider

$$\begin{tikzcd}
H \times H \arrow[r, hook, "j \times j"] \arrow[d, "*_H"'] & \bar{H} \times \bar{H} \arrow[r, hook, "\bar{i} \times \bar{i}"] \arrow[d, dashed, "*_{\bar{H}}"] & G \times G \arrow[d, "*_G"] \\
H \arrow[r, "j"] & \bar{H} \arrow[r, hook, "\bar{i}"] & G
\end{tikzcd}$$

Also

$$\begin{tikzcd}
H \arrow[r, hook, "j"] \arrow[d, "inv_H"'] & \bar{H} \arrow[r, hook, "\bar{i}"] \arrow[d, dashed] & G \arrow[d, "inv_G"] \\
H \arrow[r, "j"] & \bar{H} \arrow[r, hook, "\bar{i}"] & G
\end{tikzcd}$$

i.e. $\bar{H}$ is a subgroup of G.

NEXT CONSIDER SAME PULLBACK SQUARE:

$$\begin{tikzcd} H\times H \arrow[r, hook, "j\times Id"] \arrow[d, "\pi_1"'] & \bar{H}\times\bar{H} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \bar{H} \arrow[d, "!"] \\ H \arrow[r, hook, "j"] & \bar{H} \arrow[r, "!"] & \mathbf{1} \end{tikzcd}$$

$\Rightarrow$ $j\times Id$ DENSE ON π_2

BUT

$$\begin{tikzcd} \bar{H}\times\bar{H} \arrow[rr, "\cong"] \arrow[dr, "*_{\bar{H}}"'] & & \bar{H}\times\bar{H} \arrow[dl, "\pi_2"] \\ & \bar{H} & \end{tikzcd}$$

$$(h_1,h_2) \mapsto (h_1, h_1h_2)$$
$$(h_1, h_1^{-1}h_2) \leftarrow\!\shortmid (h_1, h_2)$$

AS $\bar{H}$ SUBGROUP $\Rightarrow$ $j\times Id$ DENSE OVER $*_{\bar{H}}$

& SYMMETRICALLY $Id\times j$ DENSE OVER $*_{\bar{H}}$

$\Rightarrow$ $j\times j$ DENSE OVER $*_{\bar{H}}$.

NEXT CONSIDER

$$\begin{tikzcd} H\times H \arrow[r, hook, "j\times j"] \arrow[d, "*_H"'] & \bar{H}\times\bar{H} \arrow[d, "*_{\bar{H}}"] \\ H \arrow[r, hook, "j"] & \bar{H} \end{tikzcd} \quad\Rightarrow\quad \begin{tikzcd} \$^{H\times H} \arrow[drr, "\exists_{*_{\bar{H}}(j\times j)}"] & & \$^{\bar{H}\times\bar{H}} \arrow[ll, "\$^{j\times j}"'] \arrow[d, "\exists_{\times_{\bar{H}}}"'] \\ \$^{H} \arrow[u, "\$^{*_H}"] & & \$^{\bar{H}} \arrow[ll, two heads, "\$j"'] \arrow[u, bend right, "\$^{*_{\bar{H}}}"'] \end{tikzcd}$$

$\Rightarrow$ $\$^{H} \cong \$^{\bar{H}}$ $\Rightarrow$ $H \cong \bar{H}$.

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ORDER DUAL.

BOTH $\underline{\mathrm{Loc}}$ & $\underline{\mathrm{Loc}}^{co}$ SATISFY THE AXIOMS.

WHAT DOES THE RESULT LOOK LIKE FOR $\underline{\mathrm{Loc}}^{co}$?

"IF $H \overset{i}{\hookrightarrow} G$ IS A SUBGROUP WITH H OPEN THEN i IS WEAKLY CLOSED IN $\underline{\mathrm{Loc}}^{co}$ "

H COMPACT IN $\underline{\mathrm{Loc}}$

IE FITTED IN $\underline{\mathrm{Loc}}$

NEW RESULT FOR LOCALE THEORY:

"EVERY COMPACT SUBGROUP IS FITTED."

SUMMARY

* IN LOCALE THEORY ALL SUBGROUPS WITH OPEN DOMAIN ARE WEAKLY CLOSED
* ORDER DUAL CATEGORICAL APPROACHES TO LOCALE THEORY EXIST.
* IN THESE APPROACHES ANY $X_0 \overset{i}{\hookrightarrow} X$ WITH X_0 OPEN FACTORS AS

$$X_0 \overset{j}{\hookrightarrow} \overline{X_0} \overset{T}{\hookrightarrow} X$$

WITH j DENSE, T WEAKLY CLOSED.

- HOFMANN-MISLOVE
- NON-STANDARD DEF$^{\underline{N}}$ OF DENSE.

* THIS ALLOWS FOR A CATEGORICAL PROOF OF THE CLOSED SUBGROUP THEOREM.
* THERE IS AN ORDER DUAL:
  - ALL COMPACT LOCALIC SUBGROUPS ARE FITTED.

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