A LOCALIC PROOF OF THE LOCALIC GROUPOID REPRESENTATION OF GROTHENDIECK TOPOSES

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Abstract. It is known that each Grothendieck topos is the category of $\mathbb{G}$-equivariant sheaves for some localic groupoid $\mathbb{G}$. A simple proof of this is given which relies on the recently observed fact that the pullback adjunction between locales induced by any geometric morphism satisfies Frobenius reciprocity.

1. Introduction

The purpose of this paper is to provide a short proof of the famous Joyal and Tierney result, Theorem 2 of 8.3 in [JT84], which shows that every Grothenedieck topos is the category of $\mathbb{G}$-equivariant sheaves for a localic groupoid $\mathbb{G}$.

We find that the following facts are needed:

1. For every geometric morphisms $f : \mathcal{F} \longrightarrow \mathcal{E}$ the induced pullback adjunction $\Sigma_f \dashv f^* : \mathbf{Loc}_{\mathcal{F}} \rightleftarrows \mathbf{Loc}_{\mathcal{E}}$ between categories of locales is stably Frobenius (i.e. satisfies Frobenius reciprocity at each slice).

2. Open localic surjections are of effective descent.

The proof relies on a categorical characterization of when an adjunction between cartesian categories arises as the natural adjunction between a cartesian category and the category of $\mathbb{G}$-objects for an internal groupoid $\mathbb{G}$ (Lemma 3.1 below). It is shown that the pullback adjunction between locales that arises from a bounded geometric morphism satisfies this characterization, given that open surjections are of effective descent. The point of the proof is that no discussion is needed of the pullback of geometric morphisms and the corresponding site charactizations of properties of geometric morphisms. Pullback of locale maps is needed but this is considerably simpler. In essence once the notion of Grothendieck topos is interpreted as bounded geometric morphism, the proof can become entirely localic.

2. Background, Notation and Definitions

In this section we introduce some background material, set notation and recall some definitions. A good source for this material is [J02] and we will mostly follow its notation. We shall use $\mathcal{F}$, $\mathcal{E}$ and $\mathcal{S}$ for arbitrary elementary toposes, and $f : \mathcal{F} \longrightarrow \mathcal{E}$ and $p : \mathcal{E} \longrightarrow \mathcal{S}$ for geometric morphisms. For any elementary topos $\mathcal{E}$ there is the cartesian category $\mathbf{Loc}_{\mathcal{E}}$ of *locales over* $\mathcal{E}$. If Y is a locale over $\mathcal{E}$ then $\Omega_{\mathcal{E}} Y$ is the corresponding frame; similarly for $f : Y \longrightarrow X$ a locale map, $\Omega_{\mathcal{E}} f : \Omega_{\mathcal{E}} X \longrightarrow \Omega_{\mathcal{E}} Y$ is the corresponding frame homomorphism.

A geometric morphism $p : \mathcal{E} \longrightarrow \mathcal{S}$ is *bounded* if there is an object B of $\mathcal{E}$ (a *bound*) with the property that for every object A of $\mathcal{E}$ there is a surjection $C \twoheadrightarrow A$ and an inclusion $C \hookrightarrow B \times p^* I$ for some objects I of $\mathcal{S}$ and C of $\mathcal{E}$. In fact this implies (and is implied by) the condition that for every locale Y over $\mathcal{E}$ there is a surjection $B \times p^* I \twoheadrightarrow \Omega_{\mathcal{E}} Y$ for some object I of $\mathcal{S}$. To see this note that any $q : C \twoheadrightarrow \Omega_{\mathcal{E}}Y$ lifts to $\overline{q} : B \times p^*I \twoheadrightarrow \Omega_{\mathcal{E}}Y$ by defining $\overline{q}(N) = \bigvee_{\Omega_{\mathcal{E}}Y}\{q(N)|N \in C\}$. In the other direction any object A of $\mathcal{E}$ embeds into the frame PA, the powerset of A. Pulling this embedding back along a surjection $B \times p^*I \twoheadrightarrow PA$ provides a subjobject of $B \times p^*I$ that maps surjectively onto A.

Given an elementary base topos $\mathcal{S}$ then another elementary topos $\mathcal{E}$ is said to be a *Grothendieck topos over* $\mathcal{S}$ provided there is a *site* $(\mathbb{C}, J)$ over $\mathcal{S}$ such that $\mathcal{E}$ is the topos of sheaves over the site; i.e., $\mathcal{E} \simeq Sh_{\mathcal{S}}(\mathbb{C}, J)$. You may consult C2 of [J02] for further detail on the definition of site and sheaves over a site. However, for this paper we do not concern ourselves with the details of sites because we use:

Theorem 2.1. *$\mathcal{E}$ is a Grothendieck topos over $\mathcal{S}$ if and only if $p : \mathcal{E} \longrightarrow \mathcal{S}$, the unique geometric morphism back to $\mathcal{S}$, is bounded.*

Proof. Theorem C2.2.8 of [J02]. □

We now set notation and give some basic background on groupoids internal to a cartesian category. Any object W of a cartesian category $\mathcal{D}$ gives rise to an internal groupoid with domain and codomain maps given by the projections $W \times W \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} W$. The groupoid multiplication is given by $\pi_{13} : W \times W \times W \longrightarrow W \times W$, the unit map is given by the diagonal $\Delta_W : W \hookrightarrow W \times W$ and the inverse map is given by the twist isomorphism $\tau : W \times W \longrightarrow W \times W$. The following lemma provides further examples of internal groupoids and in fact our main categorical lemma (Lemma 3.1 below) in effect indicates that under mild assumptions every internal groupoid arises via this class of examples:

Lemma 2.2. *If $F : \mathcal{D} \longrightarrow \mathcal{C}$ is a functor between cartesian categories and W is an object of $\mathcal{D}$ with the property that the functor $F_W : \mathcal{D}/W \longrightarrow \mathcal{C}/FW$ given by $F_W(f) = F(f)$ is cartesian, then $F(W \times W) \times_{F(W)} F(W \times W) \xrightarrow{F\pi_{13}\cong} F(W \times W) \underset{F\pi_2}{\overset{F\pi_1}{\rightrightarrows}} F(W)$ (with $F\Delta_W : F(W) \longrightarrow F(W \times W)$) is an internal groupoid of $\mathcal{C}$ with inverse given by $F(\tau) : F(W \times W) \longrightarrow F(W \times W)$. Here, $F(W \times W) \times_{F(W)} F(W \times W) \xrightarrow{\cong} F(W \times W \times W)$ is the canonical isomorphism induced by the assumption that F_W is cartesian.*

Proof. This is an exercise in the definition of groupoid. □

If $\mathcal{C}$ is a cartesian category and $\mathbb{G} \equiv (G_1 \times_{G_0} G_1 \xrightarrow{m} G_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} G_0)$ (with $s : G_0 \longrightarrow G_1$) is an internal groupoid, with say $i : G_1 \longrightarrow G_1$ the inverse map, then we use $[\mathbb{G}, \mathcal{C}]$ for the category of $\mathbb{G}$-*objects*; that is, the category whose objects are pairs $(l : X \longrightarrow G_0, \alpha : G_1 \times_{G_0} X \longrightarrow X)$ such that α satisfies the usual unit and associative conditions. Another way of looking at the same category is as the category of algebras of a monad and to describe this easily we need to clarify some notational conventions and definitions around pullbacks. If $f : X \longrightarrow Y$ is a morphism in a cartesian category we use $\Sigma_f \dashv f^* : \mathcal{C}/X \rightleftarrows \mathcal{C}/Y$ for the induced pullback adjunction. $f : X \longrightarrow Y$ is an *effective descent morphism* if the functor f^* is monadic. We will also use Z_l as notation for the morphism $l : Z \longrightarrow X$; so, for example, $\Sigma_f(Z_l) = Z_{fl}$. For the case $Y = 1$ we use $\Sigma_X \dashv X^*$ (rather than $\Sigma_{!^X} \dashv (!^X)^*$) and Z_X for the morphism $X^*Z = \pi_1 : X \times Z \longrightarrow X$ (rather than the more clumsy $(X \times Z)_{\pi_1}$).

With this notation, given a groupoid $\mathbb{G}$ internal to $\mathcal{C}$,

$\mathbb{T} \equiv (\Sigma_{d_1} d_0^* : \mathcal{C}/G_0 \longrightarrow \mathcal{C}/G_0, \eta : Id \longrightarrow \Sigma_{d_1} d_0^*, \mu : \Sigma_{d_1} d_0^* \Sigma_{d_1} d_0^* \longrightarrow \Sigma_{d_1} d_0^*)$

is a monad on $\mathcal{C}/G_0$ where $\eta_{(X_l,\alpha)} = (sl, Id)$ and $\mu_{(X_l,\alpha)} = m \times Id_X$. It can also be checked that $\mathbb{G}$-objects are the same thing as algebras for this monad and so that $(\mathcal{C}/G_0)^{\mathbb{T}} \cong [\mathbb{G}, \mathcal{C}]$. There is a $\mathbb{G}$-object $((G_1)_{d_1}, m)$ which we write as (G_1, m) and will play a central role in what follows essentially because of the following lemma:

Lemma 2.3. *For any internal groupoid $\mathbb{G}$ in a cartesian category $\mathcal{C}$*

(I) $[\mathbb{G}, \mathcal{C}]/(G_1, m) \simeq \mathcal{C}/G_0$*; and,*

(II) $!^{(G_1,m)} : (G_1, m) \longrightarrow 1$ *is an effective descent morphism.*

(I) is an easy generalization of the well known relationship $[G, \mathcal{C}]/G \cong \mathcal{C}$ for a group G internal to a cartesian category $\mathcal{C}$.

Proof. (I) For any object X_l of $\mathcal{C}/G_0$ there is a $\mathbb{G}$-object morphism $(\Sigma_{d_1} d_0^* X_l, m \times Id) \xrightarrow{\pi_1} (G_1, m)$. In the other direction send any $n : (X_l, \alpha) \longrightarrow (G_1, m)$ to the equalizer of n and sl (i.e. to the 'kernel' of n). For such n, consider $\theta : (X_l, \alpha) \longrightarrow G_1 \times_{G_0} Ker(n)$ defined by $\theta(x) = (nx, \alpha(inx, x))$; then θ^{-1} is given by $\theta^{-1}(g, x) = \alpha(g, x)$. It is also clear that $X_l \cong Ker(\pi_1 : G_1 \times_{G_0} X \longrightarrow G_1)$.

(II) It can be verified that the monad on $\mathcal{C}/G_0$ induced by the adjunction $\Sigma_{(G_1,m)} \dashv (G_1, m)^*$ under the equivalence given in (I) is equal to the monad $\mathbb{T}$ defined above. (I) then follows by the observation, already made, that $(\mathcal{C}/G_0)^{\mathbb{T}} \simeq [\mathbb{G}, \mathcal{C}]$. □

Let us now put some of these definitions to work and provide a statement of the Joyal and Tierney result for which this paper is going to provide a short proof. If $\mathbb{G}$ is a groupoid internal to $\mathbf{Loc}_{\mathcal{E}}$ for some elementary topos $\mathcal{E}$, then $B_{\mathcal{E}}\mathbb{G}$, the category of $\mathbb{G}$-*equivariant sheaves*, is the full subcategory of $[\mathbb{G}, \mathbf{Loc}_{\mathcal{E}}]$ consisting of all (X_l, α) such that l is a local homeomorphism. Recall that a locale map $f : X \longrightarrow Y$ is a *local homeomorphism* if and only if both f and the diagonal $\Delta : X \hookrightarrow X \times_Y X$ are open maps, where a locale map $f : X \longrightarrow Y$ is *open* if Ωf has a left adjoint $\exists_f$ which satisfies *Frobenius reciprocity*; that is,

$$\exists_f(a \wedge \Omega f(b)) = \exists_f(a) \wedge b$$

for each $a \in \Omega X$ and $b \in \Omega Y$. Open locale maps can be used to isolate any elementary topos $\mathcal{E}$ in $\mathbf{Loc}_{\mathcal{E}}$ as the full subcategory of discrete locales. This is because a locale X is discrete if and only if $!^X : X \longrightarrow 1$ is a local homeomorphism (Ch. 5, Section 5, Theorem 1 of [JT84]).

In essence, the Joyal and Tierney result is:

Theorem 2.4. *If $\mathcal{E}$ is a Grothendieck topos over $\mathcal{S}$, with $\mathcal{S}$ having a natural numbers object, then there exists a localic groupoid $\mathbb{G}$ over $\mathcal{S}$ such that $\mathcal{E} \simeq B_{\mathcal{S}}\mathbb{G}$.*

3. A Characterization of Adjunctions induced by Groupoids

Let $\mathbb{G}$ be an internal groupoid in a cartesian category $\mathcal{C}$. There is a functor $R_{\mathbb{G}} : \mathcal{C} \longrightarrow [\mathbb{G}, \mathcal{C}]$ which sends an object X of $\mathcal{C}$ to the $\mathbb{G}$-object $((G_0)_X, d_1 \times Id_X)$. If further $\mathcal{C}$ has coequalizers then this functor has a left adjoint, $L_{\mathbb{G}}$, that sends (X_l, α) to the coequalizer of α and $\pi_2 : G_1 \times_{G_0} X \longrightarrow X$. Note that $L_{\mathbb{G}}(G_1, m) \cong G_0$ as $d_0 : G_1 \longrightarrow G_0$ is a coequalizer of m and $\pi_2 : G_1 \times_{G_0} G_1 \longrightarrow G_1$ (it is split by s). In fact, even without the assumption that $\mathcal{C}$ has coequalizers, we must have $L_{\mathbb{G}}(G_1, m) \cong G_0$ for any functor $L_{\mathbb{G}}$ left adjoint of $R_{\mathbb{G}}$. The purpose of this section is to provide characterizations of this adjunction. The first lemma provides a general characterization and the second lemma specializes to the case of adjunctions that are stably Frobenius.

Lemma 3.1. *If $L \dashv R : \mathcal{D} \rightleftarrows \mathcal{C}$ is an adjunction between cartesian categories then the following are equivalent:*

1. There exists an object W of $\mathcal{D}$ such that $!^W : W \longrightarrow 1$ is of effective descent and the functor $L_W : \mathcal{D}/W \longrightarrow \mathcal{C}/LW$ is an equivalence.

2. There exists a groupoid $\mathbb{G}$ internal to $\mathcal{C}$ and an equivalence $\Theta : \mathcal{D} \longrightarrow [\mathbb{G}, \mathcal{C}]$ such that $\Theta R = R_{\mathbb{G}}$.

Proof. 1. implies 2. Define $\mathbb{G}$ to be the groupoid constructed from the groupoid $W \times W \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} W$ as in Lemma 2.2. So $G_0 = LW$, $G_1 = L(W \times W)$ etc. Since $W \longrightarrow 1$ is of effective descent, $\mathcal{D} \simeq (\mathcal{D}/W)^{\mathbb{T}'}$ where $\mathbb{T}'$ is the monad on $\mathcal{D}/W$ induced by the adjunction $\Sigma_W \dashv W^*$. It can be verified that under the equivalence L_W, $\mathbb{T}'$ maps to the monad $\mathbb{T}$ on $\mathcal{C}/G_0$ induced by $\mathbb{G}$. Hence $(\mathcal{D}/W)^{\mathbb{T}'} \simeq (\mathcal{C}/G_0)^{\mathbb{T}} \cong [\mathbb{G}, \mathcal{C}]$.

2. implies 1. is effectively Lemma 2.3. Observe that if $n : (X, \alpha) \longrightarrow (G_1, m)$ then $L_{\mathbb{G}}(n)$ is isomorphic $Ker(n)$ for any left adjoint $L_{\mathbb{G}}$ of $R_{\mathbb{G}}$; the inclusion of (the domain of) $Ker(n)$ is split by $x \mapsto \alpha(in(x), x)$. □

Definition 3.2. *An adjunction $L \dashv R : \mathcal{D} \rightleftarrows \mathcal{C}$ between cartesian categories*

1. satisfies Frobenius reciprocity *provided the map $L(R(X) \times W) \xrightarrow{(L\pi_1, L\pi_2)} LRX \times LW \xrightarrow{\varepsilon_X \times Id_{LW}} X \times LW$ is an isomorphism for all objects W and X of $\mathcal{D}$ and $\mathcal{C}$ respectively where ε is the conuit of the adjunction.*

2. is stably Frobenius *provided for each object X of $\mathcal{C}$ the sliced adjunction $L_X \dashv R_X : \mathcal{D}/RX \rightleftarrows \mathcal{C}/X$ given $R_X(Z_l) = RZ_{Rl}$ and $L_X(W_g) =$'the adjoint transpose of g', satisfies Frobenius reciprocity.*

An easy example of an adjunction that satisfies Frobenius reciprocity is the pullback adjunction $\Sigma_f \dashv f^*$ associated with any morphism $f : X \longrightarrow Y$ in a cartesian category. In fact it is also stably Frobenius though we shall not call on this fact in what follows.

A deeper example of a stably Frobenius adjunction is the pullback adjunction between categories of locales induced by a geometric morphism:

Example 3.3. *If $f : \mathcal{F} \longrightarrow \mathcal{E}$ is a geometric morphism between elementary toposes then there is an adjunction $\Sigma_f \dashv f^*$ between the category of locales in $\mathcal{F}$ and the category of locales in $\mathcal{E}$ with the right adjoint being given by pullback in the category of elementary toposes. $f^* : \mathbf{Loc}_{\mathcal{E}} \longrightarrow \mathbf{Loc}_{\mathcal{F}}$ can also be described using generators and relations for the theory of frames. If $\Omega_{\mathcal{E}} X \cong \mathbf{Fr}_{\mathcal{E}} < G_X | R_X >$ is a canonical presentation for a frame corresponding to a locale X over $\mathcal{E}$ (so, G_X can be taken to be $\Omega_{\mathcal{E}} X$) then the frame corresponding to $f^* X$ is presented by*

$$\mathbf{Fr}_{\mathcal{F}} < f^* G_X | f^* R_X > .$$

The left adjoint is defined by $\Omega_{\mathcal{E}} \Sigma_f X \equiv f_* \Omega_{\mathcal{F}} X$. From this it can be seen that for any locale W of $\mathcal{F}$ the frame homomorphism $\Omega_{\mathcal{F}} \eta_W$ corresponding to the unit η_W of $\Sigma_f \dashv f^*$ is the unique frame homomorphism $\Omega_{\mathcal{F}} f^* \Sigma_f W \longrightarrow \Omega_{\mathcal{F}} W$ induced by *the counit* $\varepsilon^{gm} : f^* f_* \Omega_{\mathcal{F}} W \longrightarrow \Omega_{\mathcal{F}} W$ *of the geometric morphism* f. *In other words* ε^{gm} *factors via* $\Omega_{\mathcal{F}} \eta_W$ *- a fact that we shall call on to prove our main result below.*

[T10] *shows that the adjunction* $\Sigma_f \dashv f^*$ *satisfies Frobenius reciprocity for any geometric morphism* f. *But for any locale* X *over* $\mathcal{E}$ *there is a geometric morphism* $f_X : Sh_{\mathcal{F}}(f^* X) \longrightarrow Sh_{\mathcal{E}}(X)$ *obtained by pulling back along the localic geometric morphism* $X \longrightarrow \mathcal{E}$. *Lemma 3.2 of* [T10] *confirms the easily observed fact that the pullback adjunction* $\Sigma_{f_X} \dashv (f_X)^*$ *is* $(\Sigma_f)_X \dashv (f^*)_X$ *(under* $\mathbf{Loc}_{Sh(X)} \simeq \mathbf{Loc}/X$*) and so* $\Sigma_f \dashv f^*$ *is stably Frobenius since* $(\Sigma_f)_X \dashv (f^*)_X$ *satisfies Frobenius reciprocity for each* X.

Before we specialize the previous lemma to the case that the adjunction is stably Frobenius, let us state and prove a rather trivial categorical lemma that will help the proof of said specialization:

Lemma 3.4. *Given an adjunction* $L \dashv R : \mathcal{D} \rightleftarrows \mathcal{C}$ *between cartesian categories that satisfies Frobenius reciprocity, if* $L1 \cong 1$ *and* η_W *is a regular monomorphism for each object* W *of* $\mathcal{D}$ *then the adjunction is an equivalence.*

Proof. $LRX \cong L(1 \times RX) \cong L1 \times X \cong X$; from which the counit ε is seen to be an isomorphism. For each object W there is an equalizer diagram $W \overset{\eta_W}{\hookrightarrow} RLW \underset{b}{\overset{a}{\rightrightarrows}} W'$ for some a, b and W'. Then $La = Lb$ since $L\eta_W$ is an isomorphism using the triangular identities and the fact that ε is an isomorphism. But if $La = Lb$ then they have equal adjoint transposes; that is, $\eta_{W'} a = \eta_{W'} b$ and so $a = b$ since $\eta_{W'}$ is a monomorphism. Hence η_W is an isomorphism and the adjunction in fact is an equivalence. □

Lemma 3.5. *If* $L \dashv R : \mathcal{D} \rightleftarrows \mathcal{C}$ *is a stably Frobenius adjunction between cartesian categories then the following are equivalent:*

1. There exists an object W *of* $\mathcal{D}$ *such that* $!^W : W \longrightarrow 1$ *is of effective descent and for every morphism* $f : Y \longrightarrow W$ *the morphism* $(f, \eta_Y) : Y \longrightarrow W \times_{RLW} RLY$ *is a regular monomorphism.*

2. There exists a groupoid $\mathbb{G}$ *internal to* $\mathcal{C}$ *and an equivalence* $\Theta : \mathcal{D} \longrightarrow [\mathbb{G}, \mathcal{C}]$ *such that* $\Theta R = R_{\mathbb{G}}$.

Proof. By Lemma 3.1 it is sufficient to prove that (f, η_Y) is a regular monomorphism for each $f : Y \longrightarrow W$ if and only if $L_W : \mathcal{D}/W \longrightarrow \mathcal{C}/LW$ is an equivalence. But L_W factors as $\Sigma_{\eta_W} : \mathcal{D}/W \longrightarrow \mathcal{D}/RLW$ followed by L_{LW} (i.e. the left adjoint of $L \dashv R$ sliced at LW). It has been observed already that $\Sigma_{\eta_W} \dashv \eta_W^*$ satisfies Frobenius reciprocity (as it is the pullback adjunction of a morphism in a cartesian category) and $L_{LW} \dashv R_{LW}$ satisfies Frobenius reciprocity by assumption. It is obvious that Frobenius reciprocity is closed under composition of adjunctions and so $L_W : \mathcal{D}/W \longrightarrow \mathcal{C}/LW$ is the left adjoint of an adjunction that satisfies Frobenius reciprocity. It can be seen that (f, η_Y) is the unit of this adjunction at Y_f. Hence L_W is an equivalence by application of the last lemma because $L_W(1) \cong 1$ and (f, η_Y) is a regular monomorphism for each Y_f. Conversely it is clear that if L_W is an equivalence then the unit is a regular monomorphism. □

4. A localic property of bounded geometric morphisms

Given the characterization of when an adjunction arises from a localic groupoid just given, we are clearly interested in the case of an adjunction $\Sigma_p \dashv p^*$ for some bounded geometric morphism $p : \mathcal{E} \longrightarrow \mathcal{S}$. To apply the characterization we are going to need to make a choice as to what W in $\mathbf{Loc}_{\mathcal{E}}$ is. We will take $W = [\mathbb{N} \twoheadrightarrow B]$, the locale of surjections from the natural numbers onto B, a bound for p. $\Omega_{\mathcal{E}}[\mathbb{N} \twoheadrightarrow B]$ can be presented by the generators $[i, b]$ over all $i \in \mathbb{N}$ and $b \in B$ subject to:

(I) For all $i \in \mathbb{N}$, $\bigvee_{b \in B} [i, b] = 1$.

(II) For all b_1, $b_2 \in B$ and $i \in \mathbb{N}$, $[i, b_1] \wedge [i, b_2] \leq \bigvee\{1 | b_1 = b_2\}$.

(III) For all $b \in B$, $\bigvee_{i \in \mathbb{N}} [i, b] = 1$.

(I) and (II) forces the relation on $\mathbb{N} \times B$ determined by a point of the locale being presented, to be a function. (III) forces the function to be surjection. Notice that (II) and (III) combine to imply $\bigvee_{i \in \mathbb{N}} [i, b] \wedge [i, b'] = \bigvee\{1 | b = b'\}$ for any pair b, b' of B. It can be verified, see the remarks after Proposition 2 of 5.2 in [JT84], that the locale map $! : [\mathbb{N} \twoheadrightarrow B] \longrightarrow 1$ is an open surjection and is therefore an effective descent morphism (Theorem 1, 8.2 of [JT84]).

We now state and prove a final lemma which provides an important localic property of bounded geometric morphisms. Since it has been commented already that the adjunction $\Sigma_p \dashv p^*$ is stably Frobenius, this lemma is clearly a key part of showing that bounds do indeed give rise to localic groupoids:

Lemma 4.1. *If $p : \mathcal{E} \longrightarrow \mathcal{S}$ is a bounded geometric morphism with bound B then for any locale map $f : Y \longrightarrow [\mathbb{N} \twoheadrightarrow B]$, the morphism*

$$Y \xrightarrow{(f, \eta_Y)} [\mathbb{N} \twoheadrightarrow B] \times_{\Sigma_p p^* [\mathbb{N} \twoheadrightarrow B]} \Sigma_p p^* Y$$

is a regular monomorphism.

Proof. Since B is a bound for p by an earlier remark we have that there exists I an object of $\mathcal{S}$ and a surjection $q : B \times p^* I \twoheadrightarrow \Omega_{\mathcal{E}} Y$ for any locale Y over $\mathcal{E}$. In fact, by taking the exponential and adjoint transpose of this morphism, q factors via $B \times p^* p_* (\Omega_{\mathcal{E}} Y^B) \xrightarrow{Id_B \times \varepsilon^{gm}} B \times \Omega_{\mathcal{E}} Y^B \xrightarrow{ev} \Omega_{\mathcal{E}} Y$ and so this last morphism must be a surjection, where ε^{gm} is the counit of the geometric morphism (i.e. of the adjunction $p^* \dashv p_*$). Since a locale morphism is a regular monomorphism if and only if the corresponding frame homomorphism is a surjection, it remains only to prove that $ev(Id_B \times \varepsilon^{gm}_{\Omega_{\mathcal{E}} Y^B})$ factors via $\Omega_{\mathcal{E}}(f, \eta_Y)$. For any $i \in \mathbb{N}$ define $\Theta_i : \Omega_{\mathcal{E}} Y^B \longrightarrow \Omega_{\mathcal{E}} Y$ by

$$\Theta_i(\psi) = \bigvee_{b' \in B} \psi(b') \wedge \Omega_{\mathcal{E}} f[i, b'].$$

Map $B \times p^* p_* (\Omega_{\mathcal{E}} Y^B)$ to $\Omega_{\mathcal{E}}([\mathbb{N} \twoheadrightarrow B] \times_{\Sigma_p p^* [\mathbb{N} \twoheadrightarrow B]} \Sigma_p p^* Y)$ by sending (b, N) to the join

$$\bigvee_{i \in \mathbb{N}} [i, b] \otimes g[p^* p_* (\Theta_i)](N)$$

where $g : p^*p_*\Omega_{\mathcal{E}}Y \longrightarrow \Omega_{\mathcal{E}}p^*\Sigma_pY$ is the universal map from the generators of the frame $\Omega_{\mathcal{E}}p^*\Sigma_pY$. We have observed above that $\Omega_{\mathcal{E}}\eta_Y g = \varepsilon^{gm}_{\Omega_{\mathcal{E}}Y}$. For $(b,N) \in B \times p^*p_*(\Omega_{\mathcal{E}}Y^B)$, setting $\psi \equiv \varepsilon^{gm}_{\Omega_{\mathcal{E}}Y^B}(N)$, we have

$$\begin{aligned}
\Omega_{\mathcal{E}}(f,\eta_Y)\bigvee_{i\in\mathbb{N}}[i,b]\otimes g[p^*p_*(\Theta_i)](N) &= \bigvee_{i\in\mathbb{N}}\Omega_{\mathcal{E}}f[i,b]\wedge\Omega_{\mathcal{E}}\eta_Y g[p^*p_*(\Theta_i)](N)\\
&= \bigvee_{i\in\mathbb{N}}\Omega_{\mathcal{E}}f[i,b]\wedge\varepsilon^{gm}[p^*p_*(\Theta_i)](N)\\
&= \bigvee_{i\in\mathbb{N}}\Omega_{\mathcal{E}}f[i,b]\wedge\Theta_i\varepsilon^{gm}_{\Omega_{\mathcal{E}}Y^B}(N))\\
&= \bigvee_{i\in\mathbb{N}}\Omega_{\mathcal{E}}f[i,b]\wedge\Theta_i\psi\\
&= \bigvee_{i\in\mathbb{N}}\Omega_{\mathcal{E}}f[i,b]\wedge\bigvee_{b'\in B}\psi(b')\wedge\Omega_{\mathcal{E}}f[i,b']\\
&= \bigvee_{b'\in B}\psi(b')\wedge\bigvee_{i\in\mathbb{N}}\Omega_{\mathcal{E}}f[i,b]\wedge\Omega_{\mathcal{E}}f[i,b']\\
&= \bigvee_{b'\in B}\psi(b')\wedge\bigvee\{1|b=b'\}\\
&= \psi(b)\\
&= [\varepsilon^{gm}_{\Omega_{\mathcal{E}}Y^B}(N)](b)\\
&= ev(Id_B\times\varepsilon^{gm}_{\Omega_{\mathcal{E}}Y^B})(b,N)
\end{aligned}$$

which proves that $ev(Id_B \times \varepsilon^{gm}_{\Omega_{\mathcal{E}}Y^B})$ factors via $\Omega_{\mathcal{E}}(f,\eta_Y)$ as required. □

5. Proof of the Main Theorem

Enough material has now been collected together to give a quick proof of the main theorem, Theorem 2.4.

Proof. The unique geometric morphism $p : \mathcal{E} \longrightarrow \mathcal{S}$ is bounded by Lemma 2.1. If B is a bound for p, take $W = [\mathbb{N} \twoheadrightarrow B]$ in Lemma 3.5. It can then be concluded that $\mathbf{Loc}_{\mathcal{E}} \simeq [\mathbb{G}, \mathbf{Loc}_{\mathcal{S}}]$ for some localic groupoid over $\mathcal{S}$ by application of the previous lemma. To complete the proof what is needed is a verification that under this equivalence a locale Y over $\mathcal{E}$ is discrete if an only if its image, (X_l, α) say, has that l is a local homeomorphism. Say that Y is discrete, i.e. $! : Y \longrightarrow 1$ is a local homeomophism. Under the equivalence $(\mathbf{Loc}_{\mathcal{E}}/[\mathbb{N} \twoheadrightarrow B])^{\mathbb{T}} \simeq \mathbf{Loc}_{\mathcal{E}}$, the pullback functor $[\mathbb{N} \twoheadrightarrow B]^*$ is the forgetful functor from the category of algebras of $\mathbb{T}$ that forgets the $\mathbb{T}$ action. But local homeomorphism are pullback stable and so this proves that the map X_l corresponding to Y is a local homeomorphism. In the other direction the image of (X_l, α) has $\pi_1 : [\mathbb{N} \twoheadrightarrow B] \times Y \longrightarrow [\mathbb{N} \twoheadrightarrow B]$ as the underlying object in $\mathbf{Loc}_{\mathcal{E}}/[\mathbb{N} \twoheadrightarrow B]$. π_1 is then a local homeomorphism. The result then follows by the fact (see the remarks before Theorem C5.1.6 of [J02]) that local homeomorphisms descend along open surjections. □

References

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