A categorical account of the Hofmann-Mislove theorem

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Abstract

A categorical account is given of the Hofmann-Mislove theorem, describing the Scott open filters on a frame. The account is stable under an order duality and so is shown to also cover Bunge and Funk's constructive description of the points of the lower power locale.

The categorical axioms offered are based on a representation theorem for dcpo homomorphism between frames in terms of certain natural transformations; this allows for a categorical account to be given of dcpo homomorphisms. This specializes to give a new categorical description of the upper and lower power locale constructions.

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1. *Introduction*

The Hofmann-Mislove theorem, also known as the Scott Open Filter theorem, describes an order reversing bijection between the Scott open filters of the opens of a sober topological space and the compact saturated subspaces (saturated means upper closed with respect to the specialization preorder). This result was originally shown in [**2**] and an account in textbook form is offered as Theorem 8.2.5 in [**11**]. The account in [**11**] describes the Scott open filters of an arbitrary frame in terms of compact saturated sets of points of the frame. The original result can be recovered by taking the frame to be the opens of a sober topological space since sobriety (by definition) forces the points of this frame to be in bijection with the points of the topological space.

Both accounts depend on a choice axiom, but in [**4**], Lemma 3.4, Johnstone manages to describe the Scott open filters of any frame without an assumption of choice: they are in order reversing bijection with the compact fitted sublocales. A sublocale is said to be fitted if it can be expressed as a meet of open sublocales. Notice that for a topological space a subspace is saturated if and only if it is an intersection of open subsets; thus 'fitted' is a reasonable choice for the localic notion of saturated. Therefore [**4**] provides a localic version of the Hofmann-Mislove theorem with the original result recoverable using a choice axiom.

Though Johnstone's result is free of choice, it is dependent on an ordinal induction. In [**12**] Vickers adapts the preframe techniques of [**6**] to remove the ordinal induction and (following Smyth, [**10**]) sees the Hofmann-Mislove theorem as a method of describing the points of a power construction. Recalling that the points of the upper power locale are in bijection with the Scott open filters of the frame of opens of the locale, the Hofmann-Mislove theorem is describing the points of the upper power locale. With the removal of the ordinal induction it becomes transparent that the proof of Hofmann-Mislove is constructive; that is valid in any topos. In particular it is valid in the topos of sheaves $Sh(Y)$ for any locale Y and so a description of the general points of the upper power locale becomes available by carrying out the proof relative to $Sh(Y)$ and noting that the upper power construction is stable under change of base; [**12**] provides more detail on this.

Bunge and Funk, [**1**], give a constructive proof that the points of the lower power locale are in order preserving bijection with the weakly closed sublocales that have open domain. Assuming the excluded middle all locales are open and all weakly closed sublocales are closed; from this it is routine to verify the result classically, recalling that the points of the lower power locale are in bijection with the upper closed completely coprime subsets of the frame (the complements of such subsets must be principal ideals). The constructive result describing the points of the lower power locale is also covered in [**12**]. Moreover the proofs offered there describing the points of the upper power locale (Hofmann-Mislove) and the lower power locale (Bunge-Funk) follow identical, but order dual, paths. Thus the suggestion of [**12**] is that they are, abstractly, the same result. The main purpose of this paper is to prove that they are the same result.

We achieve this by placing various categorical axioms on a category $\mathbf{C}$. The canonical model will be $\mathbf{C} = \mathbf{Loc}$, the category of locales. The first axiom is that $\mathbf{C}$ is order enriched and all the other axioms are order dual. Therefore the order dual of $\mathbf{C}$, denoted $\mathbf{C}^{co}$, will be a model if and only if $\mathbf{C}$ is a model. It will be clear from construction that the assertion

$$\text{Hofmann-Mislove true in } \mathbf{C}$$

is equivalent to

$$\text{Bunge-Funk true in } \mathbf{C}^{co}$$

and so a single abstract proof will cover both results. This axiomatic description of results in **Loc** has already been developed in [**8**] where it is used to give a categorical account of the fact that localic triquotient surjections are of effective descent. Here the same framework is adopted, but before this is set out in detail notation must be established.

2. *Background on Locales and Notation*

Familiarity with basic lattice theoretic and categorical definitions and notation is assumed (see, for example, [**3**] and [**7**]). A frame is a complete lattice which satisfies the distributivity law

$$a \wedge \bigvee T = \bigvee \{a \wedge t \mid t \in T\}$$

for any element a and subset T. A frame homomorphism preserves arbitrary joins and finite meets and so a category, **Fr**, of frames is defined which is order enriched with the obvious pointwise ordering. A frame will be denoted ΩX, where X is the *corresponding* *locale*. This comes from the definition of the category of locales:

$$\mathbf{Loc} \equiv \mathbf{Fr}^{op},$$

that is, the category of locales is taken to be the dual of the category of frames. There is therefore notational but no mathematical difference between a locale and a frame. Given a locale map $f : X \to Y$ (i.e. the localic notion of a continuous map between spaces) the corresponding frame homomorphism is denoted $\Omega f : \Omega Y \to \Omega X$. The notation $!^X : X \to 1$ is used for the unique locale map to the terminal locale 1.

Although the theory of frames contains an infinitary join operation it is suitably algebraic since coequalizers exist and free frames can be constructed, see Ch. II in [**3**]. So frame presentations present, for example the frame

$$\Omega\mathbb{S} \equiv \mathbf{Fr}\langle\{*\}\rangle$$

is well defined; that is, the free frame on a single generator. $\Omega\mathbb{S}$ can be described more concretely as the set of monotone maps $\{0 \leq 1\} \to \Omega$. Classically, therefore, $\Omega\mathbb{S}$ is the three point chain. The corresponding locale $\mathbb{S}$ is known as the Sierpiński locale, and it is an *order internal* distributive lattice in **Loc** where 'order internal' indicates that the finitary meet(join) operations are required to be right(left) adjoint to the diagonal.

A result in [**9**] shows that for any locales Y and X, dcpo homomorphisms $\Omega Y \to \Omega X$, i.e. directed join preserving maps, are in natural order isomorphism with

$$Nat[\mathbf{Loc}(\_ \times Y, \mathbb{S}), \mathbf{Loc}(\_ \times X, \mathbb{S})]$$

where $\mathbf{Loc}(\_ \times Y, \mathbb{S}) : \mathbf{Loc}^{op} \to \mathbf{Set}$ is the presheaf for any locale Y and $Nat[\_]$ is the collection of natural transformations ordered componentwise in the obvious manner. This isomorphism is an extension, to dcpo homomorphisms, of the obvious mapping:

$$\Omega f \longmapsto \mathbf{Loc}(\_ \times f, \mathbb{S})$$

for any frame homomorphism $\Omega f : \Omega Y \to \Omega X$. This representation theorem for dcpo homomorphisms in terms of natural transformations specializes. Since it can verified using Yoneda's lemma that $\mathbf{Loc}(\_ \times Y, \mathbb{S})$ is the exponential $\mathbf{Loc}(\_, \mathbb{S})^{\mathbf{Loc}(\_, Y)}$ in the presheaf category $[\mathbf{Loc}^{op}, \mathbf{Set}]$, it is clear that $\mathbf{Loc}(\_ \times Y, \mathbb{S})$ is also an internal distributive lattice in $[\mathbf{Loc}^{op}, \mathbf{Set}]$ as it inherits this structure from $\mathbb{S}$. The notation $\mathbb{S}^Y$ is used for the internal distributive lattice $\mathbf{Loc}(\_ \times Y, \mathbb{S})$. The result specializes in the following manner:

$$\mathbf{PreFr}(\Omega Y, \Omega X) \cong \sqcap - \mathbf{SLat}([\mathbf{Loc}^{op}, \mathbf{Set}])(\mathbb{S}^Y, \mathbb{S}^X)$$

$$\mathbf{Sup}(\Omega Y, \Omega X) \cong \sqcup - \mathbf{SLat}([\mathbf{Loc}^{op}, \mathbf{Set}])(\mathbb{S}^Y, \mathbb{S}^X)$$

where on the right hand side we are restricting to internal meet/join semilattice homomorphisms respectively. A preframe homomorphism (**PreFr**) preserves directed joins and finite meets and a suplattice homomorphism (**Sup**) preserves arbitrary joins.

Recall the double power locale construction, denoted $\mathbb{P}$ (see, for example, the last section of [**6**]). It is usually defined using a frame presentation:

$$\Omega\mathbb{P}X \equiv \mathbf{Fr}\langle \boxtimes a,\, a \in \Omega X \mid \vee^{\uparrow}\{\boxtimes i \mid i \in I\} = \boxtimes \vee^{\uparrow} I, \quad \forall I \subseteq^{\uparrow} \Omega X\rangle,$$

or in words

$$\Omega\mathbb{P}X \equiv \mathbf{Fr}\langle\Omega X \text{ qua dcpo}\rangle.$$

From this we have that for any locales Y and X,

$$\begin{aligned}\mathbf{Loc}(Y,\mathbb{P}X) &\cong \mathbf{dcpo}(\Omega X,\Omega Y)\\ &\cong Nat[\mathbb{S}^X,\mathbb{S}^Y]\\ &\cong Nat[\mathbf{Loc}(\_,Y)\times\mathbb{S}^X,\mathbb{S}]\end{aligned}$$

which is sufficient to prove that the double exponential $\mathbb{S}^{\mathbb{S}^X}$ in $[\mathbf{Loc}^{op},\mathbf{Set}]$ exists and is naturally isomorphism to the representable $\mathbf{Loc}(\_,\mathbb{P}X)$, i.e. the main result of [**9**]. Therefore it is immediate that $\mathbb{P}$ is the functor part of a monad on **Loc**; note that the Kleisli category of this monad is equivalent to the opposite of the category whose objects are frames and whose morphisms are dcpo homomorphisms. Given the relationship between dcpo homomorphisms and natural transformations we have that:

LEMMA 1. $\mathbf{Loc}_{\mathbb{P}}^{op}$ *is equivalent to the full subcategory of* $[\mathbf{Loc}^{op},\mathbf{Set}]$ *whose objects are of the form* $\mathbb{S}^X$*.*

Proof. The mapping

$$\Omega X \longmapsto \mathbb{S}^X$$

is the object part of a full, faithful and essentially surjective functor to the full subcategory of objects of the form $\mathbb{S}^X$. $\square$

We shall also be concerned with the upper and lower power locale constructions (see Ch. 11, [**11**]), P_U and P_L:

$$\Omega P_U X \equiv \mathbf{Fr}\langle \Omega X \text{ qua preframe}\rangle,$$

$$\Omega P_L X \equiv \mathbf{Fr}\langle \Omega X \text{ qua suplattice}\rangle.$$

Note the order preserving bijection $\mathbf{Loc}(1,P_U X)\cong\{F\subseteq\Omega X \mid F \text{ is a Scott open filter}\}$ since $F\subseteq\Omega X$ is a Scott open filter if and only if its classifying map $\chi_F:\Omega X\to\Omega$ is a preframe homomorphism. It is this observation that allows us to recoup the Hofmann-Mislove theorem via a description of the points of the upper power locale.

3. *Axioms*

The Hofmann-Mislove theorem will be developed relative to a category **C** required to satisfy the following axioms, the canonical model for the axioms being $\mathbf{C}=\mathbf{Loc}$.

Axiom 1. **C** is an order enriched category with finite limits and finite coproducts.

This is well known for $\mathbf{C}=\mathbf{Loc}$ and the next distributivity axiom is also known to be true of **Loc** (see [**3**]).

Axiom 2. For any objects X,Y and Z of **C**, $X\times(Y+Z)\cong X\times Y+X\times Z$ and $X\times 0\cong 0$.

As motivation for the next axiom it is worth looking at some facts about the initial frame Ω. For any other frame ΩX, there is a unique frame homomorphism $\Omega !^X:\Omega\to\Omega X$ given by

$$i\longmapsto \bigvee\nolimits^{\uparrow}_{\Omega X}\{0_{\Omega X}\}\cup\{1_{\Omega X}\mid 1\leq i\}.$$

From this it follows that for any function $\alpha : \Omega X \to \Omega$, (and any $c \in \Omega X$, $i \in \Omega$) the weakened Frobenius law,

$$\alpha(c) \wedge i \leq \alpha(c \wedge \Omega!^X(i))$$

holds. If further α is a dcpo homomorphism then the weakened coFrobenius law,

$$\alpha(c \vee \Omega!^X(i)) \leq \alpha(c) \vee i$$

holds. Recall that $\Omega = P\{*\}$, i.e. the power set of the singleton set, and so these facts are just basic set theory. Given the relationship between dcpo homomorphisms and natural transformations, the following axiom is true when $\mathbf{C} = \mathbf{Loc}$:

Axiom 3. There exists an order internal distributive lattice $\mathbb{S}$ such that if $\alpha : \mathbb{S}^X \to \mathbb{S}$ is a morphism in $[\mathbf{C}^{op}, \mathbf{Set}]$ then

$$\text{(i)}\ \sqcap_{\mathbb{S}} (\alpha \times Id) \sqsubseteq \alpha \sqcap_{\mathbb{S}^X} (Id \times \mathbb{S}^{!^X}) \text{ and}$$

$$\text{(ii)}\ \alpha \sqcup_{\mathbb{S}^X} (Id \times \mathbb{S}^{!^X}) \sqsubseteq \sqcup_{\mathbb{S}}(\alpha \times Id).$$

The ordering on natural transformations is the pointwise ordering and certainly $\mathbb{S}^Z$, i.e. the presheaf $\mathbf{C}(\_ \times Z, \mathbb{S})$, is an order internal distributive lattice as it inherits this structure from $\mathbb{S}$. Since there will be many different meet semilattices in play, the symbol Id is used for the identity map rather than 1.

It is possible ([**8**]) to establish the axiom by assuming that $\mathbb{S}$ behaves like a subobject classifier for both open and closed subobjects. A closed subobject is, by definition, any pullback of $0_{\mathbb{S}} : 1 \to \mathbb{S}$ and an open subobject is any pullback of $1_{\mathbb{S}} : 1 \to \mathbb{S}$; so the statement 'behaves like a subobject classifier' is referring to the uniqueness part of the definition only.

Axiom 4. The contravariant functor $\mathbb{S}^{(\text{-})} : \mathbf{C} \to [\mathbf{C}^{op}, \mathbf{Set}]$ defines, on morphisms, an order isomorphism to internal distributive lattice homomorphisms in $[\mathbf{C}^{op}, \mathbf{Set}]$.

That this holds for locales is implicit from the discussion of the previous section since a morphism $\Omega Y \to \Omega X$ is a frame homomorphisms if and only if it is both a preframe homomorphism and a suplattice homomorphism. Therefore natural transformations $\mathbb{S}^Y \to \mathbb{S}^X$ that are also internal distributive lattice homomorphisms in $[\mathbf{Loc}^{op}, \mathbf{Set}]$, are of the form $\mathbb{S}^f$ for unique $f : X \to Y$. Note that this axiom forces $\mathbb{S}^{(\text{-})}$ to reflect isomorphisms, ensuring that $\mathbb{S}$ is non-trivial.

Axiom 5. For any object X of $\mathbf{C}$, the double exponential $\mathbb{S}^{\mathbb{S}^X}$ exists in the presheaf category $[\mathbf{C}^{op}, \mathbf{Set}]$ and is representable.

For locales $\mathbb{S}^{\mathbb{S}^X} \cong \mathbb{P}X$ as discussed above; so this is true for locales, and we follow the same notation for $\mathbf{C}$. Thus $\mathbb{P}$ is the functor part of a monad on $\mathbf{C}$, the *double power monad.* We use the notation $\mathbf{C}^{op}_{\mathbb{P}}$ to denote the full subcategory of $[\mathbf{C}^{op}, \mathbf{Set}]$ consisting of objects of the form $\mathbb{S}^X$; this is reasonable as $\mathbf{C}^{op}_{\mathbb{P}}$ is (equivalent to) the opposite of the Kleisli category.

By an application of Axiom 2 we have that $\mathbf{C}^{op}_{\mathbb{P}}$ is closed under binary products since

$$\mathbb{S}^{X+Y} \cong \mathbb{S}^X \times \mathbb{S}^Y \text{ and}$$
$$\mathbb{S}^0 \cong 1,$$

for any X, Y in $\mathbf{C}$. Note that therefore $\mathbb{S}^X$ is an order internal distributive lattice in $\mathbf{C}_{\mathbb{P}}^{op}$ for any X in $\mathbf{C}$. Finally note that with Axiom 4 we have that $\mathbb{S}^{(-)}$ defines a duality between $\mathbf{C}$ and $\mathbf{DLat}(\mathbf{C}_{\mathbb{P}}^{op})$. (Therefore, abstractly, $\mathbf{C}$ takes on the role of $\mathbf{Loc}$ and $\mathbf{DLat}(\mathbf{C}_{\mathbb{P}}^{op})$ takes on the role of $\mathbf{Fr}$.)

Axiom 6. (i) $\sqcap - \mathbf{SLat}(\mathbf{C}_{\mathbb{P}}^{op})$ is Cauchy complete.
(ii) $\sqcup - \mathbf{SLat}(\mathbf{C}_{\mathbb{P}}^{op})$ is Cauchy complete.

Cauchy completion is the assertion that any idempotent map α factors as $\beta\gamma$ where $\gamma\beta$ is the identity. This forces β (respectively γ) to be the equalizer (coequalizer) of the arrows $Id_{dom(\alpha)}$ and α. Part (ii) of this axiom is the order dual of part (i). The axiom is true when $\mathbf{C} = \mathbf{Loc}$ since given a preframe homomorphism $\alpha : \Omega X \to \Omega X$ where X is an arbitrary locale and $\alpha^2 = \alpha$, the subpreframe

$$A = \{a \in \Omega X \mid \alpha(a) = a\}$$

is a frame. $0_A = \alpha(0_{\Omega X})$ and for $a, b \in A$, $a \vee_A b = \alpha(a \vee_{\Omega X} b)$. Similarly for idempotent suplattice homomorphisms between frames.

The final axiom reflects a known relationship between locale equalizers and dcpo coequalizers. This relationship is introduced as the 'double coverage theorem' in [**9**] as it is an extension of Johnstone's coverage result (II 2.11 in [**3**]).

Axiom 7. For any equalizer diagram

$$E \overset{e}{\hookrightarrow} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

in $\mathbf{C}$ the diagram

$$\mathbb{S}^X \times \mathbb{S}^X \times \mathbb{S}^Y \underset{\sqcap(Id \times \sqcup)(Id \times Id \times \mathbb{S}^g)}{\overset{\sqcap(Id \times \sqcup)(Id \times Id \times \mathbb{S}^f)}{\rightrightarrows}} \mathbb{S}^X \xrightarrow{\mathbb{S}^e} \mathbb{S}^E$$

is a coequalizer in $\mathbf{C}_{\mathbb{P}}^{op}$ where $\sqcap(Id \times \sqcup)$ is the composite

$$\mathbb{S}^X \times \mathbb{S}^X \times \mathbb{S}^X \overset{Id \times \sqcup}{\longrightarrow} \mathbb{S}^X \times \mathbb{S}^X \overset{\sqcap}{\to} \mathbb{S}^X.$$

Using the notation of this axiom note that for any object Z, precomposition with $\mathbb{S}^e$ defines a meet semilattice inclusion of $Nat[\mathbb{S}^E, \mathbb{S}^Z]$ into $Nat[\mathbb{S}^X, \mathbb{S}^Z]$. To prove that $\alpha \sqsubseteq \beta$ for $\alpha, \beta : \mathbb{S}^E \to \mathbb{S}^Z$ it is sufficient to show $\alpha\mathbb{S}^e \sqsubseteq \beta\mathbb{S}^e$; this step will be used in the proof of the main theorem.

Note that this last axiom does not break the symmetry given by the order enrichment. A short calculation using the distributivity assumption on $\mathbb{S}$ shows that the composite $\sqcup(1 \times \sqcap)$ could have been used in the place of $\sqcap(1 \times \sqcup)$. The other axioms are obviously order dual and so we have,

THEOREM 1. *If an ordered enriched category* $\mathbf{C}$ *satisfies the axioms then so does its order dual,* $\mathbf{C}^{co}$.

4. *Upper and Lower Power Monads*

The double power functor at an object X is, by definition, the object whose points are *all* natural transformations from the presheaf $\mathbb{S}^X$ to $\mathbb{S}$. Since both $\mathbb{S}^X$ and $\mathbb{S}$ are internal distributive lattices we can restrict to natural transformations that are meet semilattice homomorphisms (or join semilattice homomorphisms) and hope to recover the upper and lower power constructions.

Definition 1. *(i) For any object X, define $P_U(X)$, the* upper power object of X, *to be the intersection of the equalizers in* $\mathbf{C}$ *of*

$$\text{(a) } \mathbb{S}^{\mathbb{S}^X} \rightrightarrows \mathbb{S} \quad (\text{“preserves } 1\text{”})$$
$$\text{(b) } \mathbb{S}^{\mathbb{S}^X} \rightrightarrows \mathbb{S}^{\mathbb{S}^X \times \mathbb{S}^X} \ (\text{“preserves binary meet”}).$$

In detail, the top arrow of (a) is

$$\mathbb{S}^{\mathbb{S}^X} \to 1 \xrightarrow{1_{\mathbb{S}}} \mathbb{S}$$

and the bottom arrow of (a) is

$$\mathbb{S}^{\mathbb{S}^X} \xrightarrow{\mathbb{S}^{1_{\mathbb{S}^X}}} \mathbb{S}.$$

The top arrow of (b) is the exponential transpose of

$$\begin{aligned} \mathbb{S}^{\mathbb{S}^X} \times \mathbb{S}^X \times \mathbb{S}^X &\to \mathbb{S} \\ (\Lambda, a, b) &\longmapsto \Lambda(a \sqcap_{\mathbb{S}^X} b) \end{aligned}$$

and the bottom arrow of (b) is the exponential transpose of

$$\begin{aligned} \mathbb{S}^{\mathbb{S}^X} \times \mathbb{S}^X \times \mathbb{S}^X &\to \mathbb{S} \\ (\Lambda, a, b) &\longmapsto \Lambda(a) \sqcap_{\mathbb{S}} \Lambda(b). \end{aligned}$$

(ii) The lower power object, *$P_L(X)$, is defined identically but with* 0 *in place of* 1 *in (a) and join in place of meet in (b).*

Note that $\mathbb{S}^{\mathbb{S}^X} \cong \mathbb{P}(X)$ and $\mathbb{S}^{\mathbb{S}^X \times \mathbb{S}^X} \cong \mathbb{P}(X+X)$, the latter by the distributivity axiom (Axiom 2), and so the equalizer diagrams of this definition are within $\mathbf{C}$ as the wording of the definition assumes. So there are regular monomorphisms $i^U : P_U(X) \hookrightarrow \mathbb{P}X$ and $i^L : P_L(X) \hookrightarrow \mathbb{P}X$; using $\mathbb{S}^X \xrightarrow{\Box_X} \mathbb{S}^{P_U X}$ and $\mathbb{S}^X \xrightarrow{\Diamond_X} \mathbb{S}^{P_L X}$ for the double exponential transposes of i^U and i^L respectively it is clear that $\mathbb{S}^X \xrightarrow{\Box_X} \mathbb{S}^{P_U X}$ is an internal meet semilattice homomorphism and $\mathbb{S}^X \xrightarrow{\Diamond_X} \mathbb{S}^{P_L X}$ is an internal join semilattice homomorphism (in $\mathbf{C}^{op}_{\mathbb{P}}$). In fact using exponential transpose it is clear that,

Lemma 2. *(i) For any objects X and Z of* $\mathbf{C}$, *any internal meet semilattice homomorphism $\alpha : \mathbb{S}^X \to \mathbb{S}^Z$ in $\mathbf{C}^{op}_{\mathbb{P}}$ factors as*

$$\mathbb{S}^X \xrightarrow{\Box_X} \mathbb{S}^{P_U X} \xrightarrow{\mathbb{S}^{p_\alpha}} \mathbb{S}^Z$$

for unique $p_\alpha : Z \to P_U(X)$.

(ii) For any objects X and Z of $\mathbf{C}$, *any internal join semilattice homomorphism $\alpha : \mathbb{S}^X \to \mathbb{S}^Z$ in $\mathbf{C}^{op}_{\mathbb{P}}$ factors as*

$$\mathbb{S}^X \xrightarrow{\Diamond_X} \mathbb{S}^{P_L X} \xrightarrow{\mathbb{S}^{p_\alpha}} \mathbb{S}^Z$$

for unique $p_\alpha : Z \to P_L(X)$.

The two monads that arise are the upper and lower power monads on $\mathbf{C}$ and are denoted (P_U, η^U, μ^U) and (P_L, η^L, μ^L) respectively. For example $\eta^U_X = p_{Id_{\mathbb{S}^X}}$ using the notation of the lemma, (i), and $\mu^U_X = p_{\Box_X \Box_{P_U X}}$.

Natural transformations that are also meet semilattice homomorphisms are, when $\mathbf{C} = \mathbf{Loc}$, in bijection with preframe homomorphisms. It follows that, with part (i) of this lemma, $\mathbb{S}^{P_U X}$ is the 'free frame qua preframe' as required by the definition of upper power locale. In other words the upper power object construction, and similarly the lower power object construction, coincide with the upper and lower power locale constructions when $\mathbf{C} = \mathbf{Loc}$.

Now $\mathbb{P}X$ is an order internal distributive lattice in $\mathbf{C}$ as it inherits this structure from $\mathbb{S}$. Note that $P_U(X)$ is an order internal meet semilattice in $\mathbf{C}$: it is a sub-meetsemilattice of $\mathbb{P}X$ via i^U. It follows that for any object Z, the meet semilattice $\mathbf{C}(Z, P_U(X))$ is a sub-meet semilattice of $\mathbf{C}(Z, \mathbb{P}(X))$. Dually $\mathbf{C}(Z, P_L(X))$ is a sub-join semilattice of $\mathbf{C}(Z, \mathbb{P}(X))$. In fact:

LEMMA 3. *The assignment $\alpha \longmapsto p_\alpha$, for both parts (i) and (ii) of the previous lemma, is an order isomorphism.*

Proof. Given the preamble this amounts to checking that the bijection $\mathbf{C}(Z, \mathbb{P}(X)) \cong Nat[\mathbb{S}^X, \mathbb{S}^Z]$ is an order isomorphism. It needs to be checked that $\alpha \sqsubseteq \beta : \mathbb{S}^X \to \mathbb{S}^Z$ implies $\overline{\alpha} \sqsubseteq \overline{\beta} : Z \to \mathbb{P}(X)$ where $\overline{(\_)}$ denotes taking double exponential . This will complete the proof as the converse is immediate since $\mathbb{S}^{(\_)}$ preserves order on homsets. But $\overline{\sqcap_{\mathbb{S}^Z}(\alpha, \beta)} = \sqcap_{\mathbb{P}X}(\overline{\alpha}, \overline{\beta})$ as both these expressions share the same exponential transpose since

$$\begin{array}{ccc} Z \times \mathbb{S}^X & \stackrel{\overline{\alpha} \times Id}{\longrightarrow} & \mathbb{S}^{\mathbb{S}^X} \times \mathbb{S}^X \\ Id \times \alpha \downarrow & & \downarrow ev \\ Z \times \mathbb{S}^Z & \stackrel{ev}{\rightarrow} & \mathbb{S} \end{array}$$

commutes and similarly with β in place of α. $\square$

As an application we have $p_{\sqcap_{\mathbb{S}^Z}(\alpha,\beta)} = \sqcap_{P_U X}(p_\alpha, p_\beta)$ for any meet semilattice homomorphisms $\alpha, \beta : \mathbb{S}^X \to \mathbb{S}^Z$. Further note that $\mathbb{S}^{P_U}$ can be viewed as a functor from $\sqcap - \mathbf{SLat}(\mathbf{C}^{op}_{\mathbb{P}}) \to \mathbf{DLat}(\mathbf{C}^{op}_{\mathbb{P}}) \simeq \mathbf{C}^{op}$. It is left adjoint to the forgetful functor; on objects it sends $\mathbb{S}^X$ to $\mathbb{S}^{P_U X}$ and sends meet semilattice homomorphisms $\beta : \mathbb{S}^X \to \mathbb{S}^Y$ to $\mathbb{S}^{p_{\Box_Y \beta}}$. This functor enjoys $\mathbb{S}^{\eta^U_X} \mathbb{S}^{P_U}(\beta) = \mathbb{S}^{p_\beta}$ since both sides are distributive lattice homomorphisms that agree when precomposed with $\Box_X$.

Now if (P_U, η^U, μ^U) is a co-KZ monad (that is, if μ^U is left adjoint to $P_U \eta^U$; see for example B1.1.11 of [**5**]) then $\mathbb{S}^{\eta^U_Y} \dashv \Box_Y$ for any Y, since $Id \sqsubseteq P_U(\eta^U_Y)\mu^U$ implies $\mathbb{S}^{Id} \Box_{P_U Y} \sqsubseteq \mathbb{S}^{P_U(\eta^U_Y)\mu^U} \Box_{P_U Y}$, i.e. $\Box_{P_U Y} \sqsubseteq \Box_{P_U Y} \Box_Y \mathbb{S}^{\eta^U_Y}$ and so, by postcomposition with $\mathbb{S}^{\eta_{P_U Y}}$, $Id \sqsubseteq \Box_Y \mathbb{S}^{\eta^U_Y}$. This implies that for any meet semilattice homomorphism $\beta : \mathbb{S}^X \to \mathbb{S}^Y$, $\Box_Y \mathbb{S}^{p_\beta} \sqsupseteq \mathbb{S}^{P_U}(\beta)$ a fact which we will wish to exploit in the proof of the main theorem to follow.

The final aim of this section is therefore to show that indeed (P_U, η^U, μ^U) is a co-KZ monad, given $\mathbf{C}$ satisfying the axioms. Order dually it will be the case that (P_L, η^L, μ^L) is a KZ monad, both facts being well known when $\mathbf{C} = \mathbf{Loc}$. However to prove the main result for the paper it is possible to skip this and move straight to the next section, taking the assertion

(P_U, η^U, μ^U) is a co-KZ monad and (P_L, η^L, μ^L) is a KZ monad

as a new axiom replacing Axiom 6.

The next theorem describes a new categorical relationship between a Cauchy completeness property and the co-KZness of P_U.

THEOREM 2. *Given* **C** *satisfying the axioms except possibly Axiom 6, the following are equivalent:*
(i) (P_U, η^U, μ^U) *is a co-KZ monad,*
(ii) for any idempotent deflationary meet semilattice $\alpha : \mathbb{S}^X \to \mathbb{S}^X$, $\alpha\mathbb{S}^{p_\alpha} = \mathbb{S}^{p_\alpha}$ *and*
(iii) deflationary idempotents are split in $\sqcap - \mathbf{SLat}(\mathbf{C}^{op}_{\mathbb{P}})$.

Of course an idempotent $\alpha : \mathbb{S}^X \to \mathbb{S}^X$ is deflationary if $\alpha \sqsubseteq Id$.

Although $\sqcap - \mathbf{SLat}(\mathbf{C}^{op}_{\mathbb{P}})$ is equivalent to the opposite of the Kleisli category, $\mathbf{C}^{op}_{P_U}$, the theorem does not generalize to arbitrary monads: it is not true in general, in either direction, that a monad on an order enriched category is co-KZ if and only if deflationary idempotents split in the Kleisli category. For a counter example take the identity monad (clearly co-KZ) on any order enriched category with a non-splitting deflationary idempotent, say a single object category with a single non-trivial morphism α such that $\alpha \sqsubseteq Id$ and $\alpha^2 = \alpha$. Conversely consider the powerset monad on the category of sets with discrete ordering. This is clearly not co-KZ, but deflationary idempotents split since $\alpha \sqsubseteq Id$ implies $\alpha = Id$.

Proof. (i)$\Longrightarrow$(ii). Certainly for $\alpha : \mathbb{S}^X \to \mathbb{S}^X$ with $\alpha \sqsubseteq Id$, it is true that $\alpha\mathbb{S}^{p_\alpha} \sqsubseteq \mathbb{S}^{p_\alpha}$. It is also true, as in the preamble, that $\mathbb{S}^{\eta^U_X} \dashv \Box_X$ and $\Box_X\mathbb{S}^{p_\alpha} \sqsupseteq \mathbb{S}^{P_U}(\alpha)$, provided (P_U, η^U, μ^U) is a co-KZ monad. But since $\alpha = \alpha\alpha$,

$$\begin{aligned}\mathbb{S}^{P_U}(\alpha) &= \mathbb{S}^{P_U}(\alpha)\mathbb{S}^{P_U}(\alpha)\\ &\sqsubseteq \mathbb{S}^{P_U}(\alpha)\Box_X\mathbb{S}^{p_\alpha}\\ &= \Box_X\alpha\mathbb{S}^{p_\alpha}\end{aligned}$$

and so $\mathbb{S}^{\eta^U_X}\mathbb{S}^{P_U}(\alpha) \sqsubseteq \alpha\mathbb{S}^{p_\alpha}$ as $\mathbb{S}^{\eta^U_X} \dashv \Box_X$, i.e. $\mathbb{S}^{p_\alpha} \sqsubseteq \alpha\mathbb{S}^{p_\alpha}$.

(ii)$\Longrightarrow$(iii). Let $\alpha : \mathbb{S}^X \to \mathbb{S}^X$ be a deflationary idempotent meet semilattice homomorphism. Consider the morphisms

$$\begin{aligned}&\epsilon^{\sqcup} : \mathbb{S}^X \times \mathbb{S}^X \xrightarrow{\Box_X \times \Box_X} \mathbb{S}^{P_U X} \times \mathbb{S}^{P_U X} \xrightarrow{\sqcup} \mathbb{S}^{P_U X} \text{ and}\\ &\delta^{\sqcup} : \mathbb{S}^X \times \mathbb{S}^X \xrightarrow{\alpha \times \alpha} \mathbb{S}^X \times \mathbb{S}^X \xrightarrow{\sqcup} \mathbb{S}^X \xrightarrow{\Box_X} \mathbb{S}^{P_U X}.\end{aligned}$$

Let $E^{\sqcup} \hookrightarrow P_U X$ be the equalizer in **C** of their double exponential transposes, i.e. of

$$P_U X \underset{\overline{\delta^{\sqcup}}}{\overset{\overline{\epsilon^{\sqcup}}}{\rightrightarrows}} \mathbb{P}(X + X).$$

Note that for any distributive lattice homomorphism $\gamma : \mathbb{S}^{P_U X} \to \mathbb{S}^W$, $\gamma\mathbb{S}^{\overline{\epsilon^{\sqcup}}} = \gamma\mathbb{S}^{\overline{\delta^{\sqcup}}}$ if and only if $\gamma\epsilon^{\sqcup} = \gamma\delta^{\sqcup}$; that is $\mathbb{S}^{E^{\sqcup}}$ is the quotient 'frame' of $\mathbb{S}^{P_U X}$ universally taking $\Box_X(\alpha(a) \sqcup \alpha(b))$ to the join of $\Box_X(a)$ and $\Box_X(b)$ for all $a, b : \mathbb{S}^X$. Further consider the morphisms

$$\begin{aligned}&\epsilon^0 : 1 \xrightarrow{0_{\mathbb{S}^{P_U X}}} \mathbb{S}^{P_U X} \text{ and}\\ &\delta^0 : 1 \xrightarrow{0_{\mathbb{S}^X}} \mathbb{S}^X \xrightarrow{\Box_X} \mathbb{S}^{P_U X},\end{aligned}$$

and let $E^0 \hookrightarrow P_U X$ be the equalizer in **C** of their exponential adjoint transposes. Take $e : E \hookrightarrow P_U X$ to be the intersection of E^0 and $E^{\sqcup}$; formed by pulling back E^0 along $E^{\sqcup}$.

Since α is deflationary we have that $\alpha(0) = 0$ (and so $\mathbb{S}^{p_\alpha}\epsilon^0 = \mathbb{S}^{p_\alpha}\delta^0$) and since α is also idempotent we have that

$$\alpha(\alpha(a) \sqcup \alpha(b)) = \alpha(a) \sqcup \alpha(b)$$

for $a, b : \mathbb{S}^X$, and so $\mathbb{S}^{p_\alpha}\delta^{\sqcup} = \mathbb{S}^{p_\alpha}\epsilon^{\sqcup}$. It follows that $\mathbb{S}^{p_\alpha}$ factors as $\mathbb{S}^q\mathbb{S}^e$ for some unique $q : X \to E$. We have therefore split α as $\mathbb{S}^q\mathbb{S}^e\Box_X$ and so to complete this part of the proof it must be verified that $\mathbb{S}^e\Box_X\mathbb{S}^q = Id$.

Firstly note that from the universal characterization of $\mathbb{S}^E$, we must have that $\mathbb{S}^e\Box_X\alpha = \mathbb{S}^e\Box_X$, since $\mathbb{S}^e\Box_X(0) = 0$. But then

$$\mathbb{S}^e\Box_X\mathbb{S}^q\mathbb{S}^e\Box_X = \mathbb{S}^e\Box_X.$$

Now $\mathbb{S}^e\Box_X\mathbb{S}^q\mathbb{S}^e$, i.e. $\mathbb{S}^e\Box_X\mathbb{S}^{p_\alpha}$, is certainly a meet semilattice homomorphism, and it also must preserve 0 since $\mathbb{S}^e\Box_X(0) = 0$. Using the fact that $\alpha\mathbb{S}^{p_\alpha} = \mathbb{S}^{p_\alpha}$ we can also deduce that this map preserves binary joins:

$$\begin{array}{ccccccc}
\mathbb{S}^{P_U X} \times \mathbb{S}^{P_U X} & \xrightarrow{\mathbb{S}^{p_\alpha} \times \mathbb{S}^{p_\alpha}} & \mathbb{S}^X \times \mathbb{S}^X & \xrightarrow{\Box_X \times \Box_X} & \mathbb{S}^{P_U X} \times \mathbb{S}^{P_U X} & \xrightarrow{\mathbb{S}^e \times \mathbb{S}^e} & \mathbb{S}^E \times \mathbb{S}^E \\
\sqcup \downarrow & & \sqcup(\alpha \times \alpha) \downarrow & & \sqcup \downarrow & & \downarrow \sqcup \\
\mathbb{S}^{P_U X} & \xrightarrow{\mathbb{S}^{p_\alpha}} & \mathbb{S}^X & \xrightarrow{\Box_X} & \mathbb{S}^{P_U X} & \xrightarrow{\mathbb{S}^e} & \mathbb{S}^E
\end{array}$$

$\mathbb{S}^e\Box_X\mathbb{S}^q\mathbb{S}^e$ is therefore a distributive lattice homomorphism and since it agrees with $\mathbb{S}^e$ when precomposed with $\Box_X$ it must be the case that $\mathbb{S}^e\Box_X\mathbb{S}^q\mathbb{S}^e = \mathbb{S}^e$. Finally $\mathbb{S}^e$ is an epimorphism in $\mathbf{C}_{\mathbb{P}}^{op}$ by Axiom 7 and so $\Box_X\mathbb{S}^q\mathbb{S}^e = Id$ as required.

(iii)$\Longrightarrow$(i). Let $\gamma : \mathbb{S}^W \hookrightarrow \mathbb{S}^{P_U(X)}$ be the lax equalizer of

$$\mathbb{S}^{P_U X} \quad \begin{array}{c} \xrightarrow{Id} \\ \sqcap | \\ \xrightarrow[\Box_X \mathbb{S}^{\eta^U_X}]{} \end{array} \quad \mathbb{S}^{P_U X}.$$

This exists using (iii): take α to be

$$\mathbb{S}^X \xrightarrow{(Id, \Box_X\mathbb{S}^{\eta^U_X})} \mathbb{S}^X \times \mathbb{S}^X \xrightarrow{\sqcap} \mathbb{S}^X,$$

then $\alpha^2 = \alpha$ since $\mathbb{S}^{\eta^U_X}\Box_X = Id$ and $\Box_X$ preserves binary meet. γ can be calculated as the equalizer of $Id_{\mathbb{S}^{P_U X}}$ and α. It should be clear that $\gamma : \mathbb{S}^W \hookrightarrow \mathbb{S}^{P_U(X)}$ is an internal distributive lattice homomorphism in $\mathbf{C}_{\mathbb{P}}^{op}$. For example it is true that $0 : 1 \to \mathbb{S}^{P_U X}$ enjoys $Id0 \sqsubseteq \Box_X\mathbb{S}^{\eta^U_X}0$ and so factors via γ.

Now since $\Box_X\mathbb{S}^{\eta^U_X}\Box_X = \Box_X$ we must have that $\mathbb{S}^X \xrightarrow{\Box_X} \mathbb{S}^{P_U X}$ factors through γ, say as $\mathbb{S}^X \xrightarrow{\delta} \mathbb{S}^W \overset{\gamma}{\hookrightarrow} \mathbb{S}^{P_U(X)}$. Now $\mathbb{S}^X \xrightarrow{\delta} \mathbb{S}^W$ is an internal meet semilattice homomorphism and so factors as $\mathbb{S}^{p_\delta}\Box_X$ for some $p_\delta : W \to P_U X$. Then $\gamma\mathbb{S}^{p_\delta}\Box_X = \gamma\delta = \Box_X$ and so $\gamma\mathbb{S}^{p_\delta} = Id$ as γ is a distributive lattice homomorphism. This shows that the subobject $\mathbb{S}^W \hookrightarrow \mathbb{S}^{P_U(X)}$ is the whole of $\mathbb{S}^{P_U(X)}$. This is sufficient to show $\mathbb{S}^{\eta^U_X} \dashv \Box_X$ from which it is routine to verify that μ^U is left adjoint to $P_U\eta^U$, i.e. co-KZ monad as required. For completeness we add that $\mathbb{S}^{\mu^U_X}\Box_X = \Box_{P_U X}\Box_X$ by definition of μ^U_X and so

$$\begin{aligned}
\mathbb{S}^{\mu^U_X}\mathbb{S}^{P_U\eta^U_X}\Box_{P_U X} &= \mathbb{S}^{\mu^U_X}\Box_X\mathbb{S}^{\eta^U_X} \\
&\sqsupseteq \Box_{P_U X}
\end{aligned}$$

implying that $P_U(\eta^u_x) \circ \mu^U_X \sqsupseteq Id$ by the last lemma. Therefore μ^U_X is left adjoint to $P_U\eta^U$. $\Box$

It should be remarked that the construction of E from $\alpha : \mathbb{S}^X \to \mathbb{S}^X$ in the (ii)$\Longrightarrow$(iii) part of the proof is in fact a natural generalization of the order dual of a well known construction: forming the ideal completion locale of a poset. The ideal completion locale of a poset $(X, \leq)$ is the locale $Idl(X)$ whose frame of opens is the set of upper closed subsets of X. It is called the ideal completion locale since its points, $\mathbf{Loc}(1, Idl(X))$, are order isomorphic to the ideals of X, and its construction is key to the proof that dcpo homomorphisms between frames can be represented using natural transformations. Now for any set X there is a well known order isomorphism between relations $R \subseteq X \times X$ and suplattice homomorphisms $\alpha_R : PX \to PX$, i.e. $\alpha_R : \Omega X \to \Omega X$ treating X as a discrete locale. Under this bijection relational composition is sent to function composition and so if $\leq \subseteq X \times X$ is a partial order, $\alpha_{\leq}$ is then inflationary and idempotent. The fixed set, given by Cauchy completion of $\sqcup - \mathbf{SLat}(\mathbf{Loc}_{\mathbb{P}}^{op})$, is the set of upper closed subsets of X; in other words, working order dual to the proof, $\mathbb{S}^E$ is the frame of opens of the ideal completion locale and so $E \cong Idl(X)$. The alternative construction of E as an equalizer $E \hookrightarrow P_L X$, the order dual of which we needed in the above proof, is originally observed in [**13**]. For clarity we end with,

Theorem 3. *Given* $\mathbf{C}$ *satisfying the axioms,*
(i) (P_U, η^U, μ^U) *is a co-KZ monad and*
(ii) (P_L, η^L, μ^L) *is a KZ monad.*

Proof. (i) By Axiom 6, all idempotents split in $\sqcap - \mathbf{SLat}(\mathbf{C}_{\mathbb{P}}^{op})$ and so apply the last theorem.

(ii) Order dually all idempotents split in $\sqcup - \mathbf{SLat}(\mathbf{C}_{\mathbb{P}}^{op})$ and so apply the order dual of the last theorem. $\Box$

5. *Fitted and cofitted subobjects*

In the final sections a description of the points of the upper and lower powerobjects is given. They will be certain fitted and cofitted subobjects respectively and so for this section we define these classes of subobjects, give examples, and describe a situation where the data associated with a fitted subobject has a canonical form.

Definition 2. *A regular monomorphism in* $\mathbf{C}$, $i : X_0 \hookrightarrow X$, *is*
(a) fitted *if* $i : X_0 \hookrightarrow X$ *is the lax equalizer of a diagram* $f, g : X \rightrightarrows Y$ *universally setting* $f \sqsubseteq g$ *where* f *factors via the terminal object* 1,
(b) cofitted *if* $i : X_0 \hookrightarrow X$ *is a lax equalizer of a diagram* $f, g : X \rightrightarrows Y$ *universally setting* $g \sqsubseteq f$ *where* f *factors via the terminal object* 1.

These are clearly order dual concepts. It has not been stipulated that arbitrary lax equalizers exist in $\mathbf{C}$, though note that they will exist, as equalizers of an obvious diagram, if the codomain (Y in the definition) is an order internal meet (or join) semilattice. This will suffice for our analysis.

Lemma 4. *For* $\mathbf{C} = \mathbf{Loc}$ *we have that a sublocale is fitted/cofitted by this definition iff it is fitted/weakly closed in the usual sense.*

A sublocale X_0 is fitted if it is the meet of open sublocales in the complete poset of all sublocales. Recall that a locale map $i : X_0 \to X$ is a sublocale if and only if Ωi is a frame surjection. For any $a \in \Omega X$ there is a frame surjection $\Omega X \overset{a \wedge (\_)}{\longrightarrow} \downarrow a$ and the corresponding sublocale, denoted $a \hookrightarrow X$, is open. Notice that $\downarrow a$ can be presented by $\mathbf{Fr}\langle \Omega X$ qua frame $| 1 \leq a \rangle$, from which it is clear that $\mathbf{Fr}\langle \Omega X$ qua frame $| 1 \leq a_j,\ j \in J \rangle$ presents the intersection in the complete poset of all sublocales of the set $\{a_j \hookrightarrow X \mid j \in J\}$ of open sublocales. It follows that any fitted sublocale can be presented by a set of relations of the form $\Omega!(i) \leq a$ with $a \in \Omega X$ and $i \in \Omega$. In fact the converse is true, since having a single relation of the form $\Omega!(i) \leq a$ in a presentation is equivalent to having the set of relations $\{1 \leq a \mid i\}$.

A sublocale X_0 is weakly closed if it can be presented by a set of relations of the form $a \leq \Omega!(i)$ with $a \in \Omega X$ and $i \in \Omega$. For any $a \in \Omega X$ there is a frame surjection $\Omega X \overset{a\vee(\_)}{\longrightarrow} \uparrow a$ giving rise to a closed sublocale $\neg a \hookrightarrow X$. $\uparrow a$ can be presented by $\mathbf{Fr}\langle \Omega X$ qua frame $\mid a \leq 0\rangle$ and so closed sublocales are weakly closed. In the presence of the excluded middle, $\Omega!(i) = 0$ or 1 and so classically the weakly closed sublocales are exactly the closed sublocales. These characterizations of fitted sublocale and weakly closed are originally in [**12**].

Proof. Say X_0 is fitted by the axiomatic definition and $\mathbf{C} = \mathbf{Loc}$. Then ΩX_0 is presented by $\mathbf{Fr}\langle \Omega X$ qua frame $\mid \Omega!^X(\Omega p(b)) \leq \Omega g(b),\ b \in \Omega Y\rangle$ where $p \circ !^X$ is the factorization of f via 1. X_0 is therefore fitted in the usual sense.

Conversely say that X_0 is fitted in the usual locale theory sense. Then there exists $\chi \subseteq \Omega \times \Omega X$ such that $\Omega X_0 \cong \mathbf{Fr}\langle \Omega X$ qua frame $\mid \Omega!^X(i) \leq a,\ (i,a) \in \chi\rangle$. Let the locale D be defined by $\Omega D =$ the least subframe of $\Omega \times \Omega X$ containing χ. X_0 is then a lax equalizer of a diagram $X \rightrightarrows D$.

The weakly closed assertion is identical to prove. □

Let us show that examples exist of fitted and cofitted subobjects given an arbitrary $\mathbf{C}$ satisfying the axioms.

LEMMA 5. *An open subobject is fitted and a closed subobject is cofitted.*

Proof. An open subobject is, by definition, the pullback of $1_{\mathbb{S}} : 1 \to \mathbb{S}$, and since $1_{\mathbb{S}}$ is top, this pullback is actually a lax pullback. But this lax pullback can be re-expressed as a lax equalizer and so the result follows. In detail, the open subobject classified by $a : X \to \mathbb{S}$ is given by universally setting

$$X \overset{!^X}{\to} 1 \overset{1_{\mathbb{S}}}{\to} \mathbb{S}$$

to be less than

$$X \overset{a}{\to} \mathbb{S}.$$

Cofitted result identical. □

Capturing exactly the localic definition of compactness when $\mathbf{C} = \mathbf{Loc}$ we have that an object X of $\mathbf{C}$ is *compact* if and only if the map

$$\mathbb{S}^{!^X} : \mathbb{S} \to \mathbb{S}^X$$

has a right adjoint. The right adjoint is always denoted $\forall_{!^X}$ and, being a right adjoint, is certainly a meet semilattice homomorphism.

In the case where X_0 is compact and $i : X_0 \hookrightarrow X$ is fitted we find that there is a canonical representation of the diagram for which i is a lax equalizer.

LEMMA 6. *If $i : X_0 \hookrightarrow X$ is a fitted subobject with X_0 compact then $i : X_0 \hookrightarrow X$ is the lax equalizer universally setting $X \overset{!^X}{\to} 1 \overset{p_\alpha}{\to} P_U X$ to be less than $\eta_X^U : X \to P_U X$ where*

$1 \xrightarrow{p_\alpha} P_U X$ *is the point corresponding to the meet semilattice composition* $\mathbb{S}^X \xrightarrow{\mathbb{S}^i} \mathbb{S}^{X_0} \xrightarrow{\forall_{!^{X_0}}} \mathbb{S}$.

Proof. Say $h : Z \to X$ has the property that $p_\alpha !^X h \sqsubseteq \eta_X^U h$. Then

$$\mathbb{S}^X \xrightarrow{\mathbb{S}^i} \mathbb{S}^{X_0} \xrightarrow{\forall_{!^{X_0}}} \mathbb{S} \xrightarrow{\mathbb{S}^{!^Z}} \mathbb{S}^Z$$

is less than

$$\mathbb{S}^X \xrightarrow{\mathbb{S}^h} \mathbb{S}^Z$$

in the order enrichment. Now $i : X_0 \hookrightarrow X$ is a fitted subobject and so, by definition, there exists $p : 1 \to Y$ and $g : X \to Y$ such that i universally makes $p!^X i \sqsubseteq gi$. It follows that $p!^{X_0} \sqsubseteq gi$ and therefore $\mathbb{S}^{!^{X_0}}\mathbb{S}^p \sqsubseteq \mathbb{S}^i\mathbb{S}^g$ and so

$$\begin{aligned}
\mathbb{S}^h\mathbb{S}^{!^X}\mathbb{S}^p &= \mathbb{S}^{!^Z}\mathbb{S}^p \\
&\sqsubseteq \mathbb{S}^{!^Z}\forall_{!^{X_0}}\mathbb{S}^{!^{X_0}}\mathbb{S}^p \qquad (X_0 \text{ compact}) \\
&\sqsubseteq \mathbb{S}^{!^Z}\forall_{!^{X_0}}\mathbb{S}^i\mathbb{S}^g \\
&\sqsubseteq \mathbb{S}^h\mathbb{S}^g
\end{aligned}$$

and so, using Axiom 4, $p!^X h \sqsubseteq gh$ implying that h factors uniquely via i and therefore proving that i is the lax equalizer as required. □

6. *Hofmann-Mislove*

The main result can now proved for any category **C** satisfying the axioms.

THEOREM 4. *For any object* X, $\mathbf{C}(1, P_U X) \cong \{X_0 \hookrightarrow X \mid X_0$ *compact and* $X_0 \hookrightarrow X$ *a fitted subobject*$\}$. *The bijection is order reversing.*

The techniques used to prove this assertion are a manipulation of the techniques used in [**12**].

Proof. It is a matter of definition that $\mathbf{C}(1, P_U X) \cong \sqcap - \mathbf{SLat}(\mathbb{S}^X, \mathbb{S})$ and we have verified that this is an order isomorphism. The objective of this proof is to check that $\sqcap - \mathbf{SLat}(\mathbb{S}^X, \mathbb{S})$ is in order reversing bijective correspondence with the compact and fitted subobjects of X. Given $i : X_0 \hookrightarrow X$ a fitted subobject with X_0 compact, then there exists a meet semilattice homomorphism $\forall_{!^{X_0}} : \mathbb{S}^{X_0} \to \mathbb{S}$, this is by definition of X_0 being compact. Therefore there exists a meet semilattice homomorphism from $\mathbb{S}^X$ to $\mathbb{S}$ given by $\forall_{!^{X_0}}\mathbb{S}^i$. Let us show that this assignment reverses order. Say $i : X_0 \hookrightarrow X$ is less than or equal to $i' : X_0' \hookrightarrow X$ in $Sub(X)$; i.e. say there is $j : X_0 \to X_0'$ such that $i'j = i$. Then since $\mathbb{S}^{!^{X_0'}}\forall_{!^{X_0'}} \sqsubseteq Id$ it is true that $\mathbb{S}^j\mathbb{S}^{!^{X_0'}}\forall_{!^{X_0'}}\mathbb{S}^{i'} \sqsubseteq \mathbb{S}^j\mathbb{S}^{i'}$, i.e. $\mathbb{S}^{!^{X_0}}\forall_{!^{X_0'}}\mathbb{S}^{i'} \sqsubseteq \mathbb{S}^i$ and so

$$\forall_{!^{X_0'}}\mathbb{S}^{i'} \sqsubseteq \forall_{!^{X_0}}\mathbb{S}^i$$

by taking adjoint transpose. This shows that the assignment reverses order.

On the other hand, let $\alpha : \mathbb{S}^X \to \mathbb{S}$ be a meet semilattice homomorphism. Let $p_\alpha : 1 \to P_U X$ be the corresponding point of the upper power object. Set $X_0 \overset{i}{\hookrightarrow} X$ to be the lax equalizer which universally forces $X \xrightarrow{!^X} 1 \xrightarrow{p_\alpha} P_U X$ to be less than $\eta_X^U : X \to P_U X$. This exists since $P_U X$ is an order internal meet semilattice; $X_0 \overset{i}{\hookrightarrow} X$ is the equalizer of

$$X \underset{\sqcap(p_\alpha !^X, \eta^U_X)}{\overset{p_\alpha !^X}{\rightrightarrows}} P_U X.$$

It is clear that this assignment reverses order.

Given the canonical representation of fitted subobjects (with compact domains) given in the previous section, to complete this proof it is sufficient to verify that X_0 is compact with $\alpha = \forall_{!^{X_0}} \mathbb{S}^i$.

Now by Axiom 3,

$$\alpha \sqcup_{\mathbb{S}^X} (Id \times \mathbb{S}^{!^X}) \sqsubseteq \sqcup_{\mathbb{S}}(\alpha \times Id) \text{ and}$$
$$\sqcap_{\mathbb{S}}(\alpha \times Id) \sqsubseteq \alpha \sqcap_{\mathbb{S}^X} (Id \times \mathbb{S}^{!^X}).$$

Note that by the first inequality, evaluated at $(0_{\mathbb{S}^X} !^{\mathbb{S}^X}, \alpha) : \mathbb{S}^X \to \mathbb{S}^X \times \mathbb{S}$, we have $\alpha \mathbb{S}^{!^X} \alpha \sqsubseteq \alpha$. Further, since $\alpha(1) = 1$, by the second $Id_{\mathbb{S}} \sqsubseteq \alpha \mathbb{S}^{!^X}$. Therefore $\alpha \mathbb{S}^{!^X} \alpha = \alpha$ and since α preserves binary meet we can also conclude

$$\alpha \sqcap_{\mathbb{S}^X} (\mathbb{S}^{!^X} \alpha, Id) = \alpha. \qquad (*)$$

By Axiom 7 to construct a morphism $\forall_{!^{X_0}} : \mathbb{S}^{X_0} \to \mathbb{S}$ it needs to be verified

$$\alpha \sqcap_{\mathbb{S}^X} (Id \times \sqcup_{\mathbb{S}^X})(Id \times Id \times \mathbb{S}^{p_\alpha !^X}) = \alpha \sqcap_{\mathbb{S}^X} (Id \times \sqcup_{\mathbb{S}^X})(Id \times Id \times \mathbb{S}^{\sqcap(p_\alpha !^X, \eta^U_X)}). \; (**)$$

Notice that once $(**)$ is verified and therefore, by Axiom 7, $\forall_{!^{X_0}} : \mathbb{S}^{X_0} \to \mathbb{S}$ is constructed as the unique natural transformation such that $\alpha = \forall_{!^{X_0}} \mathbb{S}^i$, it is can be shown $\forall_{!^{X_0}}$ is right adjoint as required. In detail note that, firstly $\forall_{!^{X_0}} \mathbb{S}^{!^{X_0}} = \forall_{!^{X_0}} \mathbb{S}^i \mathbb{S}^{!^X} = \alpha \mathbb{S}^{!^X} \sqsupseteq Id$. Secondly $p_\alpha !^X i \sqsubseteq \eta^U_X i$ and so $\mathbb{S}^i \sqsupseteq \mathbb{S}^i \mathbb{S}^{!^X} \alpha$ by looking at the corresponding natural transformations. Therefore $\mathbb{S}^i \sqsupseteq \mathbb{S}^{!^{X_0}} \forall_{!^{X_0}} \mathbb{S}^i$ from which $Id \sqsupseteq \mathbb{S}^{!^{X_0}} \forall_{!^{X_0}}$, see the remarks following the introduction of Axiom 7.

It is sufficient therefore, in order to complete the proof, to verify $(**)$ and in fact this can be achieved by checking

$$\alpha \sqcup_{\mathbb{S}^X} (Id \times \mathbb{S}^{p_\alpha !^X}) \sqsubseteq \alpha \sqcup_{\mathbb{S}^X} (Id \times \mathbb{S}^{\sqcap(p_\alpha !^X, \eta^U_X)})$$

since α preserves binary meet and $\mathbb{S}^{(\_)}$ preserves order. Since $\alpha \sqcup_{\mathbb{S}^X} (Id \times \mathbb{S}^{p_\alpha !^X}) = \alpha \sqcup_{\mathbb{S}^X} (Id \times \mathbb{S}^{!^X})(Id \times \mathbb{S}^{p_\alpha})$ and $\alpha \sqcup_{\mathbb{S}^X} (Id \times \mathbb{S}^{!^X}) \sqsubseteq \sqcup_{\mathbb{S}}(\alpha \times Id)$ it is sufficient to verify that $\sqcup_{\mathbb{S}}(\alpha \times \mathbb{S}^{p_\alpha}) \sqsubseteq \alpha \sqcup_{\mathbb{S}^X} (Id \times \mathbb{S}^{\sqcap(p_\alpha !^X, \eta^U_X)})$. But certainly $\alpha \pi_1 \sqsubseteq \alpha \sqcup_{\mathbb{S}^X} (Id \times \mathbb{S}^{\sqcap(p_\alpha !^X, \eta^U_X)})$ so to complete it remains to verify that

$$\mathbb{S}^{p_\alpha} \sqsubseteq \alpha \mathbb{S}^{\sqcap(p_\alpha !^X, \eta^U_X)}.$$

Before this is proved three facts are clarified. (a) Recall, from above, that for any $\beta : \mathbb{S}^Z \to \mathbb{S}^Y$, we have

$$\Box_Y \mathbb{S}^{p_\beta} \sqsupseteq \mathbb{S}^{P_U}(\beta)$$

using $\mathbb{S}^{\eta^U_Y} \dashv \Box_Y$ where $\mathbb{S}^{P_U}$ is the functor from $\sqcap - \mathbf{SLat}(\mathbf{C}^{op}_{\mathbb{P}})$ to $\mathbf{DLat}(\mathbf{C}^{op}_{\mathbb{P}})$.

(b) Recall that using the order isomorphism $\sqcap - \mathbf{SLat}(\mathbb{S}^X, \mathbb{S}) \cong \mathbf{C}(1, P_U(X))$ we have that $\sqcap(p_\alpha !^X, \eta^U_X) = p_{\sqcap_{\mathbb{S}^X}(\mathbb{S}^{!^X} \alpha, Id)}$.

(c) Finally $\mathbb{S}^{p_\alpha} = \mathbb{S}^{p_\alpha}\mathbb{S}^{P_U}(\sqcap_{\mathbb{S}^X}(\mathbb{S}^{!^X}\alpha, Id))$, since they both are internal distributive lattice homomorphisms and agree when precomposed with $\Box_X$, use $(*)$.

So to finish the proof we have,

$$\begin{aligned}\alpha\mathbb{S}^{p_{\sqcap_{\mathbb{S}^X}(\mathbb{S}^{!^X}\alpha, Id)}} &= \mathbb{S}^{p_\alpha}\Box_X\mathbb{S}^{p_{\sqcap_{\mathbb{S}^X}(\mathbb{S}^{!^X}\alpha, Id)}}\\ &\sqsupseteq \mathbb{S}^{p_\alpha}\mathbb{S}^{P_U}(\sqcap_{\mathbb{S}^X}(\mathbb{S}^{!^X}\alpha, Id))\\ &= \mathbb{S}^{p_\alpha}.\end{aligned}$$

$\Box$

Thus we have a Hofmann-Mislove theorem even without having to assume the existence of a background set theory.

7. *Bunge-Funk*

In [**1**] the points of the lower power locale are described as exactly the weakly closed sublocales with open codomain. Although the definition of weakly closed used there is different to ours, it has been shown to be equivalent to our definition in Proposition 1.5 of [**12**]. A locale, X, is open if and only if the frame homomorphism $\Omega! : \Omega \to \Omega X$ has a left adjoint. Assuming the excluded middle all locales are open, but this statement is not true in an arbitrary topos. For the category $\mathbf{C}$ an object X is defined to be *open* if the map

$$\mathbb{S}^{!^X} : \mathbb{S} \to \mathbb{S}^X$$

has a left adjoint. The left adjoint is always denoted $\exists_{!X}$. Clearly X is open in $\mathbf{C}$ if and only if it is compact in $\mathbf{C}^{co}$. We can now recover the Bunge-Funk result for $\mathbf{C}$.

Theorem 5. *For any object X, $\mathbf{C}(1, P_L X) \cong \{X_0 \hookrightarrow X \mid X_0$ open and $X_0 \hookrightarrow X$ a cofitted subobject$\}$. The bijection is order preserving.*

Proof. The proof is order dual to the proof just given of the Hofmann-Mislove theorem, or carry out Hofmann-Mislove in $\mathbf{C}^{co}$. The assignment from cofitted subobjects with open domain to meet semilattice homomorphisms $\mathbb{S}^X \to \mathbb{S}$, which sends $X_0 \overset{i}{\hookrightarrow} X$ to $\mathbb{S}^X \xrightarrow{\mathbb{S}^i} \mathbb{S}^{X_0} \xrightarrow{\exists_{!X_0}} \mathbb{S}$, now preserves order rather than reverses it since $\exists_{!X_0}$ is a left adjoint (rather than a right adjoint). $\Box$

8. *Conclusion*

Given an axiomatic account of the category of locales, we have shown an abstract result which can be used to recover both the Hofmann-Mislove theorem and Bunge-Funk's constructive description of the points of the lower power locale. Both results follow the same categorical reasoning and are order dual. Along the way a categorical account has been given of both the upper and lower power locale constructions. They are sub-meet/join semilattices of the double power locale construction, which can itself be constructed categorically as a double exponential ([**9**]). It has been shown that if the upper(lower) Kleisli category is Cauchy complete then the upper(lower) power monad is co-KZ(KZ). Conversely if the upper(lower) power monad is co-KZ(KZ) then deflationary(inflationary) idempotents split in the upper(lower) Kleisli categories. It is this extra information about the upper and lower power constructions which appears to be key to the categorical proof of the Hofmann-Mislove/Bunge-Funk results.

It is straightforward to combine the results of this paper and [**8**] to obtain descriptions of the general points of the upper and lower power object; this covers Vickers' work in [**12**] on the general points. As further work it would be interesting to demonstrate consequences of this description of the general points. It seems, provided the axioms are slice stable, that $P_U X = \mathbb{S}^X$ for X a compact Hausdorff object, as was originally shown for locales in [**12**]. However detailed verification of this fact must await a later paper.

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