Hausdorff Systems

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Abstract

The open/proper parallel is stated. Hausdorff systems (HausSys) are introduced as the proper parallel to Vickers' continuous information systems (Infosys) [Vic93] and just as Infosys corresponds to the completely distributive lattices we prove that there is a well known class of locales which correspond to the Hausdorff systems: they are the stably locally compact locales. We proves this fact, essentially by manipulating Banaschewski and Brümmers proof that stably locally compact locales are just compact regular biframes [BB88].

Since ordered Stone locales are examples of Hausdorff systems and coherent locales are examples of stably locally compact locales it is natural to ask whether the equivalence of HausSys and stably locally compact locales is an extension of localic Priestley duality i.e. of the equivalence between ordered Stone locales and coherent locales [Tow97]. Some work needs to be done on defining the maps between Hausdorff systems before we can be sure of this.

1 Introduction

A poset is called a *frame* if and only if it has all joins, finite meets and arbitary joins distribute over finite meets. A frame homomorphsim preserves all joins and finite meets. We define the category of locales (Loc) to be the opposite of the category Frm of frames. The category of locales can be treated as if it is a category of topological spaces [Joh82]. If $f : X \to Y$ is a locale map then we write $\Omega f : \Omega Y \to \Omega X$ for the corresponding frame homomorphism. f is said to be *open* if Ωf has a left adjoint $\exists_f$ which is a SUP-lattice homomorphism and this adjoint pair satisfies the Frobenius condition

$$\exists_f(a \wedge \Omega f(b)) = \exists_f(a) \wedge b \qquad \forall a \in \Omega X, b \in \Omega Y$$

Spatially we are thinking of the open continuous functions. The open locale maps were investigated in Joyal and Tierney [JT84] where they show that for any locale X, X is discrete (i.e. $\Omega X = PA$ for some set A) if and only if

$$\begin{array}{ccc} X & \xrightarrow{!} & 1 \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

are both open.
We say $f : X \to Y$ is *proper* if and only if Ωf has a right adjoint $\forall_f$ which is a preframe homomorphism and $\forall a \in \Omega X, b \in \Omega Y$,

$$\forall_f(a \vee \Omega f(b)) = \forall_f(a) \vee b$$

Spatially we are thinking of the proper continuous functions. A locale X is compact regular if and only if it is compact and for every $a \in \Omega X$ we have that

$$a = \bigvee^{\uparrow}\{b | b \lhd a\}$$

where $b \lhd a$ iff there exists $c \in \Omega X$ such that $b \wedge c = 0$ and $a \vee c = 1$. Spatially the compact regular locales are the well known compact Hausdorff spaces. It can be shown ([Ver91], [Tow96a]) that a locale X is compact regular if and only if

$$X \xrightarrow{!} 1$$
$$X \xrightarrow{\Delta} X \times X$$

are both proper. Consequently it is natural to refer to the compact Hausdorff locales when we are talking about the compact regular locales.

It should now appear reasonable to make the statement 'compact Hausdorff locales are parallel to discrete locales' given that the definition of open map is 'parallel' to the definition of proper map. This parallel is investigated in [Tow96a]. There is a pair of parallel results which we will need in this paper. The open result can be checked as an easy exercise:

Lemma 1.1 *There is a bijection between relations on a pair of sets X, Y (i.e. subsets of $X \times Y$) and SUP-lattice homomorphisms from PX to PY. Furthermore relational composition coresponds to function compostion under this bijection.* □

It is not quite clear that this can be viewed as a statement about locales. However a little thought leads to the following restating:

Lemma 1.2 *There is a bijection between the open sublocales of $X \times Y$ for dicrete X, Y and SUP-lattice homomorphisms from ΩX to ΩY. Relational composition is sent to function composition under this bijection.*

The proper parallel is a little harder to prove (see Chapter 5 of [Tow96a] for details, or [Vic95] for a concise proof). Before we state it recall that it is a well known fact that the category of compact Hausdorff locales is regular and so a relational compostion of closed relations on compact Hausdorff locales can be defined.

Lemma 1.3 *There is a bijection between the closed sublocales of $X \times Y$ for compact Hausdorff X, Y and preframe homomorphisms from ΩX to ΩY. Relational composition is sent to function composition under this bijection.*

In [Vic93] Vickers introduces the category of continuous information systems (**Infosys**). These are pairs (X, R) where X is a set and R is a relation on X which is idempotent with respect to relational composition. There are many morphisms possible between continuous information systems. The most general are relations:

$$R : (X, R_X) \to (Y, R_Y)$$

$R \subseteq X \times Y$ such that $R = R_Y \circ R \circ R_X$ where $\circ$ is relational composition. These are called the lower semicontinuous approximable mappings.

So it is natural to introduce the parallel of continuous information systems as follows: a Hausdorff system is a pair (X, R) where X is a compact Hausdorff locale and R is a closed relation such that $R \circ R = R$. If (X, R) is an infosys, then we know from lemma 1.1 that there is a SUP-lattice homomorphism $\downarrow^R : PX \to PX$ corresponding to R. $\downarrow^R$ is idempotent since R is. The set

$$\{T | T \in PX \quad \downarrow^R T = T\}$$

can then easily be seen to be a completely distributive lattice. The essence of [Vic93] is a proof that all completely distributive lattices arise in this way. Given a Hausdorff system (X, R) we know that there is a preframe morphism $\Downarrow^{op} : \Omega X \to \Omega X$ corresponding to R which is idempotent (Lemma 1.3). Hence

$$\{a | a \in \Omega X \quad \Downarrow^{op} a = a\}$$

is a subpreframe of ΩX. It has finite joins: $\Downarrow^{op} 0$ is least and the join of a, b is given by $\Downarrow^{op} (a \vee b)$. Further,

Lemma 1.4 $\Omega \bar{X} \equiv \{a | a \in \Omega X \quad \Downarrow^{op} a = a\}$ *is a stably locally compact locale.*

Although it is well known what the class of completely distributive lattices is (for instance we can view them spatially as just the continuous posets), it is less well known what the stably locally compact locales are. Banschewski and Brümmer describe them as corresponding to the 'most reasonable not necessarily compact Hausdorff spaces'. Johnstone ([Joh82]) captures them as exactly the restracts in **Loc** of the coherent locales. The standard definition is a locale whose frame of opens is a stably continuous lattice, i.e. the directed join map from the ideal completion of the frame to itself has a left adjoint, and that left adjoint preserves finite meets. Or, to put this another way, every open is the join of opens way below ($\ll$) it, and (i) $1 \ll 1$, (ii) $a \ll b_1, b_2$ implies $a \ll b_1 \wedge b_2$.

Proof of lemma: First we check that the frame is continuous i.e. that $\forall a \in \Omega \bar{X}$

$$a = \bigvee^{\uparrow} \{b | b \ll_{\Omega \bar{X}} a\} \qquad (*)$$

Since ΩX is compact regular we know that $(\forall a, b \in \Omega X)$

$$a \lhd b \quad \Leftrightarrow \quad a \ll b$$

hence to conlude (*) all we need do is check that

$$b \ll a \quad \Rightarrow \quad \Downarrow^{op} b \ll_{\Omega \bar{X}} a$$

if $a \in \Omega \bar{X}$. Say $b \ll a$ and $a \leq \bigvee^{\uparrow} S \quad S \subseteq^{\uparrow} \Omega \bar{X}$ then $\exists s \in S \quad b \leq s \quad \Rightarrow \Downarrow^{op} b \leq \Downarrow^{op} s = s$.
As for stability we need to check that $1 \ll_{\Omega \bar{X}} 1$ (trivial by compactness of ΩX) and $a \ll_{\Omega \bar{X}} b_1, b_2$ implies $a \ll_{\Omega \bar{X}} b_1 \wedge b_2$. Since $b_i \in \Omega \bar{X}$, ΩX is regular and $\Downarrow^{op}$ is a preframe homomorphism we know that

$$b_i = \bigvee^{\uparrow} \{\Downarrow^{op} c | c \lhd b_i\}$$

Hence $a \leq \Downarrow^{op} c_i$ for some c_1, c_2 with $c_i \lhd b_i$. Hence $a \leq \Downarrow^{op} (c_1 \wedge c_2)$. But $c_1 \wedge c_2 \lhd b_1 \wedge b_2$ and so $c_1 \wedge c_2 \ll b_1 \wedge b_2$. Hence $a \ll_{\Omega \bar{X}} b_1 \wedge b_2$. $\quad \Box$.

The main aim of this paper is to prove that every stably locally compact locale arises in this way. The proof is essentially a manipulation of Banaschewski and Brümmer's proof that stably locally compact locales are dual to compact regular biframes [BB88].

2 Stably locally compact locales

Bearing in mind the correspondence between preframe homorphisms on the frame of opens of compact Hausdorff locales and closed relations on these locales then, provided we generalize our definition of morphism in **StLocKLoc** to include all preframe homomorphisms it should be clear that there is a functor:

$$\begin{array}{rcl} \mathcal{C} : \mathsf{HausSys} & \rightarrow & \mathsf{StLocKLoc} \\ (X, R) & \mapsto & \bar{X} \end{array}$$

where $\Omega \bar{X} = \{a \in \Omega X | \Downarrow^{op} a = a\}$.
We want to define

$\mathcal{B}$: StLocKLoc $\to$ HausSys

Say X is a stably locally compact locale. Define $\Lambda\Omega X$ to be the set of Scott open filters of ΩX. So $F \in \Lambda\Omega X$ iff

$$\text{(i) } F \text{ is upper}$$
$$\text{(ii) } a, b \in F \quad \Rightarrow \quad a \wedge b \in F$$
$$\text{(iii) } 1 \in F$$
$$\text{(iv) } a \in F \quad \Rightarrow \quad \exists b \in F \quad b \ll a$$

Lemma 2.1 *$\Lambda\Omega X$ is a stably locally compact locale* $\square$.

Now since X is stably locally compact we know that there is a frame injection $\downarrow : \Omega X \to Idl\Omega X$. Now define $B_{\Omega X}$ to be the free Boolean algebra on ΩX qua distributive lattice. There is a frame injection of $Idl\Omega X$ into $IdlB_{\Omega X}$ which we will denote by Ωl. So if we compose this injection with $\downarrow$ we find that ΩX can be embedded in $IdlB_{\Omega X}$.

Lemma 2.2 *$\Lambda\Omega X$ can be embedded into $IdlB_{\Omega X}$.*

Proof: Send F to $\bigvee^{\uparrow}_{b \in F} \downarrow \neg b$. It is routine to check that this is a frame injection. $\square$

Define: Y=the subframe of $IdlB_{\Omega X}$ generated by the image of the above two embeddings.

Theorem 2.1 *Y is a compact Hausdorff locale.*

Proof: Compactness is immediate since ΩY is a subsframe of the compact frame $IdlB_{\Omega X}$. As for regularity it is clearly sufficient to check that

$$\Omega l{\downarrow}a = \bigvee{}^{\uparrow}\{I | I \lhd \Omega l{\downarrow}a\}$$

for every $a \in \Omega X$ and

$$\bigvee{}^{\uparrow}_{b \in F} \downarrow \neg b = \bigvee{}^{\uparrow}\{I | I \lhd \bigvee{}^{\uparrow}_{b \in F} \downarrow \neg b\}$$

$\forall F \in \Lambda\Omega X$.

However $a = \bigvee^{\uparrow}\{x | x \ll a\}$ and $F = \bigvee^{\uparrow}\{G | G \ll F\}$ since both ΩX and $\Lambda\Omega X$ are continuous posets. So it is sufficient to prove that

$$x \ll a \quad \Rightarrow \quad \Omega l{\downarrow}x \lhd \Omega l{\downarrow}a \qquad \text{(I)}$$

$$G \ll F \quad \Rightarrow \quad \bigvee^{\uparrow}_{b \in G} \downarrow \neg b \lhd \bigvee^{\uparrow}_{b \in F} \downarrow \neg b \qquad \text{(II)}$$

(I): Say $x \ll a$. Set $F = \uparrow x$ (a Scott open filter). Then $\bigvee^{\uparrow}_{b \in F} \downarrow \neg b \in \Omega Y$. But clearly

$$\Omega l{\downarrow}x \wedge \bigvee{}^{\uparrow}_{b \in F} \downarrow \neg b = 0$$

Further $x \ll a \quad \Rightarrow \quad \exists \bar{a} \quad x \ll \bar{a} \ll a$. Hence

$$\begin{aligned} \Omega l{\downarrow}a \vee \bigvee^{\uparrow}_{b \in F} \downarrow \neg b \quad &\geq \quad \Omega l{\downarrow}a \vee \downarrow \bar{a} \\ &\geq \quad \downarrow \bar{a} \vee \downarrow \neg \bar{a} = 1 \end{aligned}$$

Hence $\Omega l{\downarrow}x \lhd \Omega l{\downarrow}a$

(II): Say $G \ll F$. So $\exists a \in F \quad G \subseteq \uparrow x \subseteq F$ (since $F = \bigvee^{\uparrow}\{\uparrow x | x \in F\}$). Then

$$\bigvee{}^{\uparrow}_{b \in G} \downarrow \neg b \wedge \Omega l{\downarrow}x = 0$$

Now $x \in F \quad \Rightarrow \quad \exists \bar{x} \in F \quad \bar{x} \ll x$ and so

$$\Omega l{\downarrow}x \vee \bigvee{}^{\uparrow}_{b \in F} \downarrow \neg b \geq \downarrow \bar{x} \vee \downarrow \neg \bar{x} = 1 \qquad \square$$

We want a closed idempotent relation on Y and so we need to find a preframe endomorphism $\epsilon : \Omega Y \to \Omega Y$ such that $\epsilon^2 = \epsilon$. If $I, J \in \Omega Y$ we write $I \bar{\lhd} J$ if and only if $\exists F \in \Lambda\Omega X$ such that

$$I \wedge \bigvee^{\uparrow}_{b\in F} \downarrow \neg b = 0$$
$$J \vee \bigvee^{\uparrow}_{b\in F} \downarrow \neg b = 1$$

Clearly $\bar{\lhd} \subseteq \lhd$ and the last proof has shown us that $x \ll a$ implies $\Omega l{\downarrow\!\!\!\downarrow}x \bar{\lhd} \Omega l{\downarrow\!\!\!\downarrow}x \lhd \Omega l{\downarrow\!\!\!\downarrow}a$. Define

$$\begin{array}{rcl} \epsilon : \Omega Y & \rightarrow & \Omega Y \\ J & \mapsto & \bigvee^{\uparrow}\{I | I = \downarrow{\downarrow\!\!\!\downarrow}a \text{ some } a, I \bar{\lhd} J\} \end{array}$$

<u>Facts about ϵ:</u>

⋆ $\forall J, \quad \epsilon(J) = \downarrow{\downarrow\!\!\!\downarrow}a$ for some $a \in \Omega X$

⋆ $\epsilon(\Omega l{\downarrow\!\!\!\downarrow}a) = \Omega l{\downarrow\!\!\!\downarrow}a \quad \forall a$

⋆ $\epsilon^2 = \epsilon$

⋆ ϵ is a preframe homomorphism.

Hence define $\mathcal{B} : \mathsf{StLocKLoc} \rightarrow \mathsf{HausSys}$ by $\mathcal{B}(X) = (Y, R_\epsilon)$, where R_ϵ is the closed relation corresponding to ϵ. It should be quite clear from contruction that $\mathcal{CB}(X) \cong X$.

3 Further Work

A lot more needs to be done to pad out this result. Essentially the morphisms are still missing. Given the results of Vickers', [Vic93], we hope to find three different classes of morphisms between Hausdorff systems that correspond to three different classes of morphisms between stably locally compact locales. The most general morphism have been implied by the work so far. They are the upper approximable semimappings.

In 'Localic Priestley Duality' [Tow97] we see that the category of coherent locales is equivalent to the category of ordered Stone locales. These are pairs

$$(X, \leq)$$

Where X is a Stone locale and $\leq$ is a closed partially order on X (which also satisfies an 'ordered Stone locale' condition). So ordered Stone locales are examples of Hausdorff systems and coherent locales are easily seen to be stably locally compact. The Hausdorff system result thus seems to extend localic Priestley duality. However we need to realise that the maps between coherent locales are the coherent maps i.e. some care has to be taken to be sure what maps we are looking at between Hausdorff systems before we can properly conclude that the equivalence between ordered Stone locales and coherent locales is infact a restriction of the equivalence between Hausdorff systems and stably locally compact locales.

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