

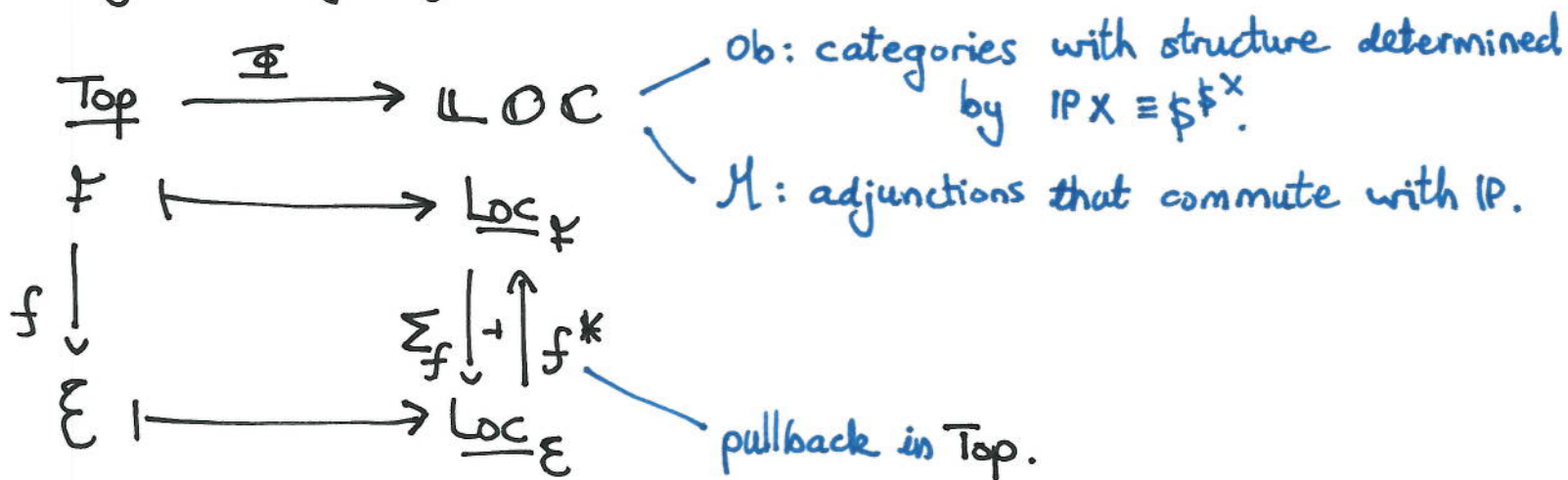
Geometric Morphisms as structure preserving maps.

CHRISTOPHER TOWNSEND. CT2014. CAMBRIDGE.

Talk idea: explain the sentence

"... re-interpreting toposes as categories of locales we see that geometric morphisms can ~~be~~ re-interpreted as structure preserving maps."

Specifically \mathbb{F} is full and faithful. Where:



Notation: dcpo = directed complete posets. dcpo_ε = dcpo relative to an elementary topos ε.

If C = order enriched finite products, DLAT_C = order internal distributive lattices. Fr ≅ DLAT(dcpo).

Loc ≅ Fr^{op}; f: X → Y locale map, then f*: 0Y → 0X frame hom. There is an adjunction

dcpo $\xrightleftharpoons[u]{F}$ Fr defining a comonad on Fr. IP is the corresponding monad on Loc.



Ob(Loc) are order enriched cartesian categories C such that:

① C has finite coproducts and ∀ f: X → Y f*: C/Y → C/X preserves finite coproduct.

② ∃ \$ ∈ Ob(DLAT_C) s.t.

(i) [X $\xrightarrow[a_2]{a_1}$ \$ & a₁* (1_{\$}) ≅ a₂* (1_{\$})] ⇒ a₁ = a₂

(ii) [X $\xrightarrow[a_2]{a_1}$ \$ & a₁* (0_{\$}) ≅ a₂* (0_{\$})] ⇒ a₁ = a₂

i.e. 1 $\xrightarrow{!}$ \$, 1 $\xrightarrow{0}$ \$
classify something.

open/closed sublocales when C = Loc.

③ ∀ X ∈ Ob(C) the exponential C(-, \$)^{\$^X} exists in [C^{op}, Set] and is representable.

Here \$^X = C(- × X, \$) = C(-, \$)^{C(-, X)} (or y \$^{y^X}). Defines IPX ≅ \$ \$^X, a monad.

Key properties of IP if it exists:

easy

full subcat.

$$\forall Y \quad \mathcal{C}(Y, IPX) \cong \underline{\text{Nat}}[\$^X, \$^Y]; \quad \mathcal{C}_{IP}^{op} \cong \{ \$^X : X \in \text{Ob}(\mathcal{C}) \} \subseteq [\mathcal{C}^{op}, \underline{\text{Set}}].$$

Def of LOC ctd.

$$\textcircled{4} \quad E \xrightarrow{f} X \xrightarrow{g} Y \text{ equalizer in } \mathcal{C} \Rightarrow \$^X \times \$^X \times \$^Y \xrightarrow[\cap(1 \times \cup)(1d \times 1d \times \$^f)]{\cap(1 \times \cup)(1d \times 1d \times \$^f)} \$^X \xrightarrow{\$^E} \$^E \text{ coequalizer in } \mathcal{C}_{IP}^{op}$$

⑤ $\$^{(-)}$ reflects isos.

⑥ Define $\mathcal{C}_\cup^{op} = \{ \mathcal{C}_{IP}^{op}, \text{morphisms } \cup\text{-slat homs.} \}$
 $\mathcal{C}_\cap^{op} = \{ \mathcal{C}_{IP}^{op}, \text{morphisms } \cap\text{-slat homs.} \}$ } \mathcal{C}_\cup^{op} and \mathcal{C}_\cap^{op} Cauchy complete.

Examples of Ob(LOC): Loc_Σ for any elementary topos Σ.

$$\underline{\text{Loc}}(Y, IPX) \cong \underline{\text{dcpo}}(O_X, O_Y) \cong \underline{\text{Nat}}[\$^X, \$^Y]$$

Vickers/Townsend

Are the Axioms any good?

- * can construct P_0, P_2 upper, lower power monads.
- * Hofmann-Mislove theorem holds.
- * Prove closed subgroup theorem.
- * slice stable.
- * Recover fundamental theorem of toposes.
- * Patch construction
- * Proper/open surjections are of effective descent.

i.e. the diagram commutes in $[C^o, \text{Set}]$.

Example axiomatic definition.

$f: X \rightarrow Y$ open if $\exists \exists_f: \$^X \rightarrow \$^Y \exists_f \dashv \f and $\exists_f(a \cap \$^f b) = \exists_f a \cap b$.

$$\underline{\text{DIS}}_{\mathcal{C}} = \{ X \in \text{Ob}(\mathcal{C}) \mid !: X \rightarrow 1 \text{ and } X \xrightarrow{\Delta} X \times X \text{ open} \} \hookrightarrow \mathcal{C}$$

full subcat.

Taylor: if this has a right adjoint then $\text{DIS}_{\mathcal{C}}$ is a topos.

Morphisms of $\underline{\text{Loc}}$ are adjunctions $\mathcal{D} \begin{matrix} \xrightarrow{L} \\ \xleftarrow{R} \end{matrix} \mathcal{E}$ s.t. $R(0) \cong 0, R(x+y) \cong R(x)+R(y)$

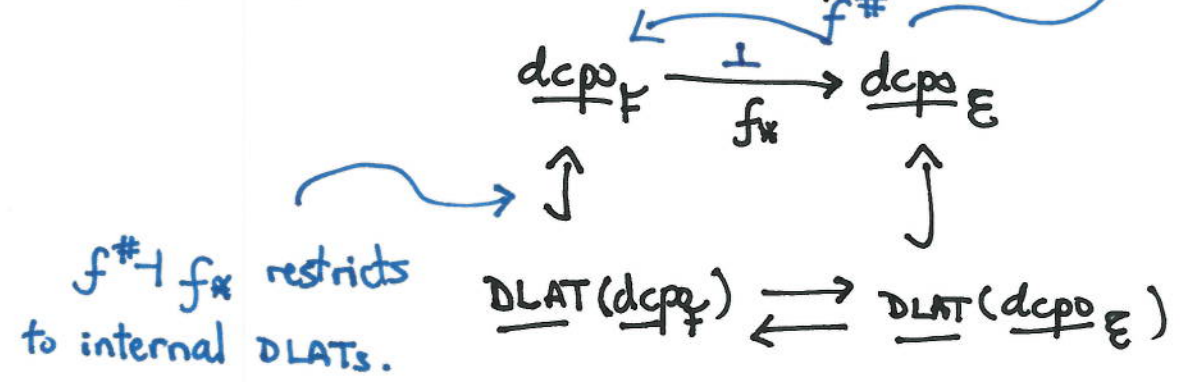
and $\exists \phi: R \circ \mathbb{P}_{\mathcal{E}} \xrightarrow{\cong} \mathbb{P}_{\mathcal{D}} \circ R$ a monad isomorphism: i.e. $L \dashv R$ is a structure preserving map.

To define $\Phi: \underline{\text{Top}} \rightarrow \underline{\text{Loc}}$ on morphisms say we are given a geometric morphism $f: \mathcal{F} \rightarrow \mathcal{E}$
 $\mathcal{E} \mapsto \underline{\text{Loc}}_{\mathcal{E}}$

Its direct image $f_*: \mathcal{F} \rightarrow \mathcal{E}$ preserves dcpos: defined by

$$f^* \underline{\text{dcpo}}_{\mathcal{E}} \langle G|R \rangle \equiv \underline{\text{dcpo}}_{\mathcal{F}} \langle f^*G | f^*R \rangle$$

where G, R are dcpo generators and relations.



i.e. (via op) $\underline{\text{Loc}}_{\mathcal{F}} \begin{matrix} \xrightarrow{\Sigma_f} \\ \xleftarrow{f^*} \end{matrix} \underline{\text{Loc}}_{\mathcal{E}}$; f^* preserves coproduct.

$$\text{Sh}_{\mathcal{F}}(f^*X) \rightarrow \text{Sh}_{\mathcal{E}}(X) \text{ p.b. of toposes.}$$

$$\downarrow \quad \downarrow$$

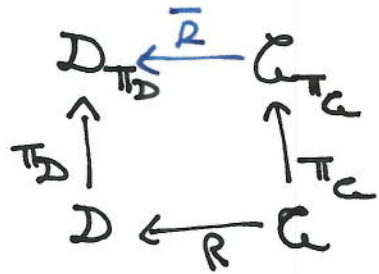
$$\mathcal{F} \xrightarrow{f} \mathcal{E}$$

So: $\Phi(f) \equiv \Sigma_f \dashv f^*$.

The inverse image of f can be recovered from $\Sigma_f + f^*$ as f^* preserves discrete locales. ⑥
 $\Rightarrow \underline{\underline{\Phi}}$ faithful ($\& \Sigma \hookrightarrow \underline{\text{Loc}}_\Sigma$ as discrete locales)

Is $\Sigma_f + f^*$ a morphism of LOC ? Does it preserve structure?

Lemma [STREET 72, PUMPLÜN 70] Given categories with monads $(\mathcal{C}, \pi_{\mathcal{C}})$, $(\mathcal{D}, \pi_{\mathcal{D}})$ and $R: \mathcal{C} \rightarrow \mathcal{D}$

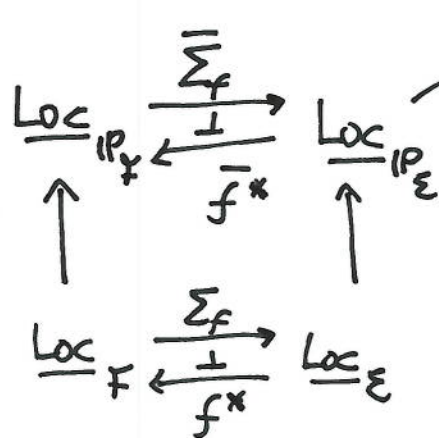


\exists a monad opfunctor $R T_{\mathcal{C}} \xrightarrow{\phi} T_{\mathcal{D}} R$

$$\Leftrightarrow \exists \bar{R}: \mathcal{C}_{\pi_{\mathcal{C}}} \rightarrow \mathcal{D}_{\pi_{\mathcal{D}}} \quad \bar{R} \pi_{\mathcal{C}} \cong \pi_{\mathcal{D}} R.$$

[This extends to left adjoints: $L \dashv R$ lifts to $\bar{L} \dashv \bar{R}$ iff $\exists \phi$ an isomorphism.]

Application



$\{ \text{ob.: frames, } \mathcal{M}: \text{dcpo homs} \}^{\text{op}}$; the lift exists by construction $f^\# + f^*$

$$\Rightarrow \exists \phi: f^* \mathbb{P}_\Sigma \xrightarrow{\cong} \mathbb{P}_F f^*$$

So: $\Sigma_f + f^* \in \mathcal{M}(\text{LOC})$.

Proof sketch that Φ is full.

We are given $L \dashv R: \underline{Loc}_F \xrightleftharpoons[\underline{R}]{\underline{L}} \underline{Loc}_E$, R preserves finite coproduct. $\exists \phi: R \mathbb{P}_E \xrightarrow{\cong} \mathbb{P}_F R$.

So \forall locales X, Y in E there is a lifting: $\underline{dcpo}_E(\mathcal{O}X, \mathcal{O}Y) \xrightarrow{\bar{R}^{op}} \underline{dcpo}_F(\mathcal{O}RX, \mathcal{O}RY)$
 $\cup^{op} \quad \cup^{op}$
 $\underline{Loc}_E(Y, X) \xrightarrow{R} \underline{Loc}_F(RY, RX)$

Similarly \bar{L}^{op} exists lifting L ; e.g. $\underline{dcpo}_F(\mathcal{O}W_1, \mathcal{O}W_2) \xrightarrow{\bar{L}^{op}} \underline{dcpo}_E(\mathcal{O}LW_1, \mathcal{O}LW_2) \quad \forall$ locales W_1, W_2 over F .

Now say $f: X \rightarrow Y$ open then Rf is open because $\bar{R}^{op}(\exists_f) = \exists_{Rf}$

But X discrete $\Leftrightarrow !: X \rightarrow 1$ & $X \xrightarrow{\Delta} X \times X$ open $\Rightarrow !^{RX}$ & Δ_{RX} open $\Rightarrow RX$ discrete.

Hence R restricts to $\underline{DIS}_E \xrightarrow{\quad} \underline{DIS}_F$ allowing us to define $f_R^*: E \rightarrow F$; cartesian as finite limits in E are created in \underline{Loc}_E .

Now for any $A \in \text{Ob}(F)$ [i.e. any set] $A \xrightarrow{\exists} \mathbb{P}_F A$ is an equalizer of

$$\begin{array}{ccc} \mathbb{I} & \longrightarrow & (\mathbb{I} \times \mathbb{I}, \{*\}) \\ \mathbb{P}_F A & \xrightarrow[\psi_2]{\psi_1} & \mathbb{P}_F(A \times A) \times \Omega_F \end{array} \quad \begin{array}{l} \text{opens of } A \times A + 1 \\ \end{array}$$

$$\mathbb{I} \longrightarrow (\{(\iota_i, i) \mid i \in \mathbb{I}\}, \exists * \in \mathbb{I})$$

ψ_1, ψ_2 are dcpo homs. so define $(f_R)_* A$ to be the equalizer of $\bar{L}^{op}(\mathbb{P}_F A) \xrightleftharpoons[\bar{L}^{op}\psi_2]{\bar{L}^{op}\psi_1} \bar{L}^{op}(\mathbb{P}_F(A \times A) \times \Omega_F)$

Proof \mathbb{E} full ctd. Say $B \in \text{Ob}(\mathcal{E})$.

(8)

$$\mathbb{E}(B, (f_R)_* A) \hookrightarrow \mathbb{E}(B, \bar{\Gamma}^{\text{op}}(P_{\mathbb{F}} A))$$

$$\cong \text{Sup}_{\mathcal{E}}(P_{\mathcal{E}} B, \bar{\Gamma}^{\text{op}}(P_{\mathbb{F}} A))$$

$$\cong \text{Sup}_{\mathbb{F}}(\bar{R}^{\text{op}} P_{\mathcal{E}} B, P_{\mathbb{F}} A)$$

$$\cong \text{Sup}_{\mathbb{F}}(P_{\mathbb{F}} f_R^* B, P_{\mathbb{F}} A)$$

$P_{\mathcal{E}} B = \text{free suplattice on } B$

as R preserves finite coproduct, $\bar{R}^{\text{op}} | \bar{\Gamma}^{\text{op}}$ specialises to suplattice homomorphisms.

$$F(f_R^* B, A) \hookrightarrow F(f_R^* B, P_{\mathbb{F}} A)$$

You then 'check naturality' $\textcircled{*}$ for ψ 's to conclude: $\mathbb{E}(B, (f_R)_* A) \cong F(f_R^* B, A)$

i.e. $f_R^* \dashv (f_R)_*$ $\Rightarrow f_R$ is a geometric morphism. □

$\textcircled{*}$ See paper for detail. It requires a bit more than is being inferred here.