# Geometric Morphisms as structure preserving maps.

CHRISTOPHER TOWNSEND. CT2014. CAMBRIDGE.

Talk idea: explain the sentence

"... re-intepreting toposes as categories of locales we see that geometric morphisms can ~~be~~ be re-interpreted as structure preserving maps."

Specifically $\Phi$ is full and faithful. Where:

$$
\begin{tikzcd}
\underline{\mathrm{Top}} \arrow[r, "\Phi"] & \mathbb{LOC} \\
\mathcal{F} \arrow[d, "f"'] \arrow[r, mapsto] & \underline{\mathrm{Loc}}_{\mathcal{F}} \arrow[d, shift right, "\Sigma_f"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{E} \arrow[r, mapsto] & \underline{\mathrm{Loc}}_{\mathcal{E}} \arrow[u, shift right, "f^*"']
\end{tikzcd}
$$

Ob: categories with structure determined by $\mathbb{P}X \equiv \$^{\$^X}$.

$\mathcal{M}$: adjunctions that commute with $\mathbb{P}$.

pullback in Top.

Notation: $\underline{\text{dcpo}}$ = directed complete posets. $\underline{\text{dcpo}}_{\mathcal{E}}$ = dcpos relative to an elementary topos $\mathcal{E}$.

If $\mathcal{C}$ = order enriched finite products, $\underline{\text{DLAT}}_{\mathcal{C}}$ = order internal distributive lattices. $\underline{\text{Fr}} \cong \underline{\text{DLAT}}(\underline{\text{dcpo}})$.

$\underline{\text{Loc}} \equiv \underline{\text{Fr}}^{op}$ ; $f: X \to Y$ locale map, then $f^*: \mathcal{O}Y \to \mathcal{O}X$ frame hom. There is an adjunction

$$\underline{\text{dcpo}} \underset{U}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \underline{\text{Fr}}$$

defining a comonad on $\underline{\text{Fr}}$. $\mathbb{P}$ is the corresponding monad on $\underline{\text{Loc}}$.

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$Ob(\mathbb{LOC})$ are order enriched cartesian categories $\mathcal{C}$ such that:

① $\mathcal{C}$ has finite coproducts and $\forall f: X \to Y$ $f^*: \mathcal{C}/Y \to \mathcal{C}/X$ preserves finite coproduct.

② $\exists \$ \in Ob(\underline{\text{DLAT}}_{\mathcal{C}})$ s.t.

(i) $[\, X \overset{a_1}{\underset{a_2}{\rightrightarrows}} \$ \;\&\; a_1^*(1_\$) \cong a_2^*(1_\$)\,] \Rightarrow a_1 = a_2$

(ii) $[\, X \overset{a_1}{\underset{a_2}{\rightrightarrows}} \$ \;\&\; a_1^*(0_\$) \cong a_2^*(0_\$)\,] \Rightarrow a_1 = a_2$

i.e. $1 \xrightarrow{1_\$} \$$, $1 \xrightarrow{0_\$} \$$ classify something.

open/closed sublocales when $\mathcal{C} = \underline{\text{Loc}}$.

③ $\forall X \in Ob(\mathcal{C})$ the exponential $\mathcal{C}(-,\$)^{\$^X}$ exists in $[\mathcal{C}^{op}, \underline{\text{Set}}]$ and is representable.

Here $\$^X = \mathcal{C}(- \times X, \$) = \mathcal{C}(-,\$)^{\mathcal{C}(-,X)}$ (or $y\$^{yX}$). Defines $\mathbb{P}X \equiv \$^{\$^X}$, a monad.

Key properties of $\mathbb{P}$ if it exists: easy

$\forall Y \quad \mathcal{C}(Y, \mathbb{P}X) \cong \underline{\mathrm{Nat}}[\$^X, \$^Y]$; $\quad \mathcal{C}^{op}_{\mathbb{P}} \cong \{\$^X : X \in \mathrm{Ob}(\mathcal{C})\} \subseteq [\mathcal{C}^{op}, \underline{\mathrm{Set}}]$. full subcat.

Def$^{\underline{n}}$ of $\amalg\mathbb{D}\mathbb{C}$ ctd.

④ $E \xhookrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ equalizer in $\mathcal{C}$ $\Rightarrow$ $\$^X \times \$^X \times \$^Y \underset{\sqcap(1\times\amalg)(\mathrm{Id}\times\mathrm{Id}\times\$^g)}{\overset{\sqcap(1\times\amalg)(\mathrm{Id}\times\mathrm{Id}\times\$^f)}{\rightrightarrows}} \$^X \xrightarrow{\$^e} \$^E$ coequalizer in $\mathcal{C}^{op}_{\mathbb{P}}$

⑤ $\$^{(-)}$ reflects isos.

⑥ Define $\left.\begin{array}{l} \mathcal{C}^{op}_{L} = \{\mathcal{C}^{op}_{\mathbb{P}}, \text{ morphisms } \amalg\text{-slat homs.}\} \\ \mathcal{C}^{op}_{U} = \{\mathcal{C}^{op}_{\mathbb{P}}, \text{ morphisms } \sqcap\text{-slat homs.}\} \end{array}\right\}$ $\mathcal{C}^{op}_{U}$ and $\mathcal{C}^{op}_{L}$ Cauchy complete.

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Examples of $\mathrm{Ob}(\amalg\mathbb{D}\mathbb{C})$: $\underline{\mathrm{Loc}}_{\mathcal{E}}$ for any elementary topos $\mathcal{E}$.

$\underline{\mathrm{Loc}}(Y, \mathbb{P}X) \cong \underline{\mathrm{dcpo}}(\mathcal{O}X, \mathcal{O}Y) \cong \underline{\mathrm{Nat}}[\$^X, \$^Y]$

Vickers/Townsend

## Are the Axioms any good?

* can construct $P_U, P_L$ upper, lower power monads.
* Hofmann-Mislove theorem holds.
* Prove closed subgroup theorem.
* slice stable.
* Recover fundamental theorem of toposes.
* Patch construction
* Proper/open surjections are of effective descent.

⋮

## Example axiomatic definition.

i.e. the diagram commutes in $[\mathcal{C}^{op}, \underline{Set}]$.

$f : X \longrightarrow Y$ <u>open</u> if $\exists$ $\exists_f : \$^X \longrightarrow \$^Y$ $\exists_f \dashv \$^f$ and $\exists_f(a \sqcap \$^f b) = \exists_f a \sqcap b$.

full subcat.

$\underline{DIS}_{\mathcal{C}} = \{ X \in Ob(\mathcal{C}) \mid ! : X \to 1 \text{ and } X \xrightarrow{\Delta} X \times X \text{ open} \} \hookrightarrow \mathcal{C}$

<u>Taylor</u>: if this has a right adjoint then $DIS_{\mathcal{C}}$ is a topos.

Morphisms of $\mathbb{LOC}$ are adjunctions $\mathcal{D} \underset{R}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathcal{C}$ s.t. $R(0) \cong 0$, $R(X+Y) \cong R(X) + R(Y)$

and $\exists\, \phi : R\mathbb{P}_{\mathcal{C}} \xrightarrow{\cong} \mathbb{P}_{\mathcal{D}} R$ a monad isomorphism: i.e. $L \dashv R$ is a structure preserving map.

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To define $\Phi : \underline{\mathrm{Top}} \longrightarrow \mathbb{LOC}$, $\mathcal{E} \longmapsto \underline{\mathrm{Loc}}_{\mathcal{E}}$ on morphisms say we are given a geometric morphism $f : \mathcal{F} \to \mathcal{E}$

Its direct image $f_* : \mathcal{F} \to \mathcal{E}$ preserves dcpos:

$$\begin{array}{ccc} \underline{\mathrm{dcpo}}_{\mathcal{F}} & \underset{f_*}{\overset{\perp}{\longrightarrow}} & \underline{\mathrm{dcpo}}_{\mathcal{E}} \\ \uparrow & & \uparrow \\ \underline{\mathrm{DLAT}}(\underline{\mathrm{dcpo}}_{\mathcal{F}}) & \rightleftarrows & \underline{\mathrm{DLAT}}(\underline{\mathrm{dcpo}}_{\mathcal{E}}) \end{array}$$

$f^{\#}$ defined by $f^{\#}\underline{\mathrm{dcpo}}_{\mathcal{E}}\langle G \mid R \rangle \equiv \underline{\mathrm{dcpo}}_{\mathcal{F}}\langle f^*G \mid f^*R \rangle$ where $G, R$ are dcpo generators and relations.

$f^{\#} \dashv f_*$ restricts to internal DLATs.

i.e. (via op) $\underline{\mathrm{Loc}}_{\mathcal{F}} \underset{f^*}{\overset{\Sigma_f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \underline{\mathrm{Loc}}_{\mathcal{E}}$ ; $f^*$ preserves coproduct.

$$\begin{array}{ccc} Sh_{\mathcal{F}}(f^*X) & \longrightarrow & Sh_{\mathcal{E}}(X) \\ \downarrow & & \downarrow \\ \mathcal{F} & \xrightarrow{f} & \mathcal{E} \end{array}$$ p.b. of toposes.

So: $\Phi(f) \equiv \Sigma_f \dashv f^*$.

The inverse image of $f$ can be recovered from $\Sigma_f \dashv f^*$ as $f^*$ preserves discrete locales.

$\Rightarrow$ $\Phi$ faithful (& $\mathcal{E} \hookrightarrow \underline{Loc}_{\mathcal{E}}$ as discrete locales)

Is $\Sigma_f \dashv f^*$ a morphism of $\mathbb{LOC}$? Does it preserve structure?

**Lemma** [STREET 72, PUMPLÜN70] Given categories with monads $(\mathcal{C}, \mathbb{T}_{\mathcal{C}})$, $(\mathcal{D}, \mathbb{T}_{\mathcal{D}})$ and $R : \mathcal{C} \to \mathcal{D}$

$$\begin{tikzcd} \mathcal{D}_{\mathbb{T}_{\mathcal{D}}} & \mathcal{C}_{\mathbb{T}_{\mathcal{C}}} \arrow[l, "\bar{R}"'] \\ \mathcal{D} \arrow[u, "\mathbb{T}_{\mathcal{D}}"] & \mathcal{C} \arrow[l, "R"] \arrow[u, "\mathbb{T}_{\mathcal{C}}"'] \end{tikzcd}$$

$\exists$ a monad opfunctor $R\mathbb{T}_{\mathcal{C}} \xrightarrow{\phi} \mathbb{T}_{\mathcal{D}} R$

$\Leftrightarrow \exists \bar{R} : \mathcal{C}_{\mathbb{T}_{\mathcal{C}}} \to \mathcal{D}_{\mathbb{T}_{\mathcal{D}}}$ $\quad \bar{R}\mathbb{T}_{\mathcal{C}} \cong \mathbb{T}_{\mathcal{D}} R$.

[This extends to left adjoints: $L \dashv R$ lifts to $\bar{L} \dashv \bar{R}$ iff $\exists \phi$ an isomorphism.]

**Application**

$$\begin{tikzcd} \underline{Loc}_{\mathbb{P}_{\mathcal{F}}} \arrow[r, shift left, "\bar{\Sigma}_f"] \arrow[r, phantom, "\perp"] & \underline{Loc}_{\mathbb{P}_{\mathcal{E}}} \arrow[l, shift left, "\bar{f}^*"] \\ \underline{Loc}_{\mathcal{F}} \arrow[u] \arrow[r, shift left, "\Sigma_f"] \arrow[r, phantom, "\perp"] & \underline{Loc}_{\mathcal{E}} \arrow[u] \arrow[l, shift left, "f^*"] \end{tikzcd}$$

$\{\text{ob.: frames}, \mathcal{M}: \text{dcpo homs}\}^{op}$; the lift exists by construction $f^{\#} \dashv f_*$

$\Rightarrow \exists \phi : f^* \mathbb{P}_{\mathcal{E}} \xrightarrow{\cong} \mathbb{P}_{\mathcal{F}} f^*$

<u>So:</u> $\Sigma_f \dashv f^* \in \mathcal{M}(\mathbb{LOC})$.

## Proof sketch that $\Phi$ is full. (7)

We are given $L \dashv R : \underline{\mathrm{Loc}}_{\mathcal{F}} \underset{R}{\overset{L}{\rightleftarrows}} \underline{\mathrm{Loc}}_{\mathcal{E}}$, $R$ preserves finite coproduct. $\exists\, \phi : R\mathbb{P}_{\mathcal{E}} \xrightarrow{\cong} \mathbb{P}_{\mathcal{F}} R$.

So $\forall$ locales $X, Y$ in $\mathcal{E}$ there is a lifting:

$$\begin{array}{ccc} \underline{\mathrm{dcpo}}_{\mathcal{E}}(\mathcal{O}X, \mathcal{O}Y) & \xrightarrow{\overline{R}^{op}} & \underline{\mathrm{dcpo}}_{\mathcal{F}}(\mathcal{O}RX, \mathcal{O}RY) \\ \cup\!|^{op} & & \cup\!|^{op} \\ \underline{\mathrm{Loc}}_{\mathcal{E}}(Y, X) & \xrightarrow{R} & \underline{\mathrm{Loc}}_{\mathcal{F}}(RY, RX) \end{array}$$

Similarly $\overline{L}^{op}$ exists lifting $L$; e.g. $\underline{\mathrm{dcpo}}_{\mathcal{F}}(\mathcal{O}W_1, \mathcal{O}W_2) \xrightarrow{\overline{L}^{op}} \underline{\mathrm{dcpo}}_{\mathcal{E}}(\mathcal{O}LW_1, \mathcal{O}LW_2)$ $\forall$ locales $W_1, W_2$ over $\mathcal{F}$.

Now say $f : X \to Y$ open then $Rf$ is open because $\overline{R}^{op}(\exists_f) = \exists_{Rf}$

**But** $X$ discrete $\Leftrightarrow$ $! : X \to 1$ & $X \xrightarrow{\Delta} X \times X$ open $\Rightarrow$ $!^{RX}$ & $\Delta_{RX}$ open $\Rightarrow$ $RX$ discrete.

Hence $R$ restricts to $\mathrm{DIS}_{\mathcal{E}} \longrightarrow \mathrm{DIS}_{\mathcal{F}}$ allowing us to define $f_R^* : \mathcal{E} \to \mathcal{F}$; cartesian

$$\begin{array}{ccc} \mathrm{DIS}_{\mathcal{E}} & \longrightarrow & \mathrm{DIS}_{\mathcal{F}} \\ \cong & & \cong \\ \mathcal{E} & \xrightarrow{f_R^*} & \mathcal{F} \end{array}$$

as finite limits in $\mathcal{E}$ are created in $\underline{\mathrm{Loc}}_{\mathcal{E}}$.

Now for any $A \in Ob(\mathcal{F})$ [i.e. any set] $A \xrightarrow{\{\cdot\}} P_{\mathcal{F}} A$ is an equalizer of

$$\begin{array}{ccc} I & \longmapsto & (I \times I, \{*\}) \\ P_{\mathcal{F}} A & \underset{\psi_2}{\overset{\psi_1}{\rightrightarrows}} & P_{\mathcal{F}}(A \times A) \times \Omega_{\mathcal{F}} \\ I & \longmapsto & (\{(i,i) \mid i \in I\}, \exists * \in I) \end{array}$$

opens of $A \times A + 1$

$\psi_1, \psi_2$ are dcpo homs. so define $(f_R)_* A$ to be the equalizer of $\overline{L}^{op}(P_{\mathcal{F}} A) \underset{\overline{L}^{op}\psi_2}{\overset{\overline{L}^{op}\psi_1}{\rightrightarrows}} \overline{L}^{op}(P_{\mathcal{F}}(A \times A) \times \Omega_{\mathcal{F}})$

Proof $\Phi$ full ctd. Say $B \in Ob(\mathcal{E})$.

$$\begin{array}{ccc}
\mathcal{E}(B,(f_R)_*A) & \hookrightarrow & \mathcal{E}(B,\overline{L}^{op}(P_{\mathcal{F}}A)) \\
 & & \cong \\
 & & \underline{Sup}_{\mathcal{E}}(P_{\mathcal{E}}B,\overline{L}^{op}(P_{\mathcal{F}}A)) \\
 & & \cong \\
 & & \underline{Sup}_{\mathcal{F}}(\overline{R}^{op}P_{\mathcal{E}}B, P_{\mathcal{F}}A) \\
 & & \cong \\
 & & \underline{Sup}_{\mathcal{F}}(P_{\mathcal{F}}f_R^*B, P_{\mathcal{F}}A) \\
 & & \cong \\
\mathcal{F}(f_R^*B, A) & \hookrightarrow & \mathcal{F}(f_R^*B, P_{\mathcal{F}}A)
\end{array}$$

$P_{\mathcal{E}}B$ = free suplattice on $B$

as $R$ preserves finite coproduct, $\overline{R}^{op} \dashv \overline{L}^{op}$ specialises to suplattice homomorphisms.

You then 'check naturality'* for $\psi_i$'s to conclude: $\mathcal{E}(B,(f_R)_*A) \cong \mathcal{F}(f_R^*B, A)$

i.e. $f_R^* \dashv (f_R)_*$ $\Rightarrow$ $f_R$ is a geometric morphism. $\square$

* See paper for detail. It requires a bit more than is being inferred here.