A representation theorem for geometric morphisms

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Abstract

It it shown that geometric morphisms between elementary toposes can be represented as certain adjunctions between the corresponding categories of locales. These adjunctions are characterized by (i) they preserve the order enrichment and the Sierpiński locale, and (ii) they satisfy Frobenius reciprocity.

1 Introduction

Given a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ between elementary toposes there is an adjunction $\Sigma_f \dashv f^* : \mathbf{Loc}_{\mathcal{F}} \rightleftarrows \mathbf{Loc}_{\mathcal{E}}$ between the category of locales in $\mathcal{F}$ and the category of locales in $\mathcal{E}$. The right adjoint of this adjunction, $f^* : \mathbf{Loc}_{\mathcal{E}} \to \mathbf{Loc}_{\mathcal{F}}$, is given by pullback in the category of toposes; see [J02] where the notation $f_!$ is used in place of Σ_f. If one replaces toposes with the categories of locales, the question arises as to whether we can offer a categorical characterization of all the adjunctions $\Sigma_f \dashv f^*$? This would show us what the correct notion of geometric morphism should be in such a context. It is this idea of identifying the correct notion of geometric morphism in the context where toposes are replaced with their categories of locales that motivates the work.

Our first result is that the adjunction $\Sigma_f \dashv f^*$ satisfies the Frobenius reciprocity condition. It is well known that this adjunction is order enriched and that f^* preserves the Sierpiński locale. Our main result is to show that, conversely, for any order enriched adjunction $L \dashv R : \mathbf{Loc}_{\mathcal{F}} \rightleftarrows \mathbf{Loc}_{\mathcal{E}}$ such that R preserves the Sierpiński locale and for which Frobenius reciprocity holds, there exists a geometric morphism $f : \mathcal{F} \to \mathcal{E}$, unique up to natural isomorphism, such that $R \cong f^*$ (and so also, $L \cong \Sigma_f$). This provides a representation theorem for geometric morphisms.

2 Outline contents

The next section recalls some results from locale theory and topos theory, essentially outlining how the adjunction $\Sigma_f \dashv f^* : \mathbf{Loc}_{\mathcal{F}} \rightleftarrows \mathbf{Loc}_{\mathcal{E}}$ is constructed for any geometric morphism $f : \mathcal{F} \to \mathcal{E}$. The representation of dcpo homomorphisms between frames as natural transformations is recalled and we also recall a technical step in the proof of this representation that is required for our main result.

The following section then proves that $\Sigma_f \dashv f^*$ always satisfies Frobenius reciprocity by application of a known Beck-Chevalley result in topos theory. The main result of the paper then follows which consists of a proof that, conversely, this provides a characterization of when such an adjunction arises from a geometric morphism.

Let us summarise the main argument which amounts to constructing a geometric morphism given an order enriched adjunction $L \dashv R : \mathbf{Loc}_{\mathcal{F}} \rightleftarrows \mathbf{Loc}_{\mathcal{E}}$ satisfying Frobenius reciprocity and with R preserving the Sierpiński locale. In broad terms we make the following observations. Firstly such an adjunction extends contravariantly to dcpo homomorphisms. This is because dcpo homomorphisms between frames can be represented as natural transformations and it is this categorical interpretation of dcpo homomorphisms that combines with the Frobenius reciprocity condition to allow the required extension. Next we observe that the existence of this extension implies that R preserves discrete locales since the property of being discrete can be characterized in terms of dcpo homomorphisms. Since the objects of any topos occur as the discrete locales in its internal category of locales we have a candidate for the inverse image of a geometric morphism $f : \mathcal{F} \to \mathcal{E}$. To obtain its direct image we observe that every object of a topos occurs as an equalizer of dcpo homomorphisms between frames. Since the direct image, if it exists, must preserve such an equalizer, it is clear how we should define the direct image. It then becomes routine to check that we have indeed defined a geometric morphism.

3 Topos theory background

Let A be a complete lattice, then A is a *frame* provided

$$\bigvee\{a \wedge t \mid t \in T\} = a \wedge \bigvee T$$

for any subset $T \subseteq A$ and any element $a \in A$. For example, the opens of a topological space form a frame. A frame homomorphism is a map between frames that preserves arbitrary joins ($\bigvee$) and finite meets ($\wedge$). For example the inverse image of any continuous map between topological spaces defines a frame homomorphism. If a complete lattice is a frame the notation ΩX is used, where X is known as the *corresponding locale*. This comes from the definition of the category of locales:

$$\mathbf{Loc} \equiv \mathbf{Fr}^{op}$$

where **Fr** is the category of frames. The notation for a frame homomorphism from ΩX to ΩY is Ωf where $f : Y \to X$ is the corresponding locale map. The category of locales is generally considered to be a reasonable category in which to do topological space theory. Probably the nicest categorical aspect of locales is that it is stable under slicing in the sense that

$$\mathbf{Loc}/X \simeq \mathbf{Loc}_{Sh(X)}; \qquad \text{Equation 1}$$

that is, the category of locales internal to the category of sheaves over a locale X is equivalent to the category of locales sliced at X ([JT84]). This is not true of the category of topological spaces, but neatly extends the well known topological relationship

$$\mathbf{LH}/X \simeq Sh(X);$$

where **LH** is the category of topological spaces with local homeomorphisms as morphisms. The category $Sh(X)$ embeds in $\mathbf{Loc}_{Sh(X)}$ as the full subcategory of discrete locales where, of course, a locale Y is discrete if and only if $\Omega Y \cong PA$ for some set A, where PA denotes the power set[1] on A. The complete lattice PA is the free *suplattice* on A, where a suplattice homomorphism is required to preserve arbitrary joins.

The property of being discrete can be characterized using open maps. A locale map $f : X \to Y$ is *open* provided Ωf has a left adjoint $\exists_f$ for which the Frobenius condition holds; that is,

$$\exists_f (a \wedge \Omega f(b)) = \exists_f(a) \wedge b$$

for any $a \in \Omega X$ and $b \in \Omega Y$. It can be shown that a locale map is open if and only if the direct image of any open sublocale is open, so this definition is well motivated. Further, under mild separation axioms the usual topological notion is recovered, see the remarks before lemma C1.5.3 of [J02] for details. It can be shown that a locale is discrete if and only if both the unique map $! : X \to 1$ and the diagonal $\Delta_X : X \hookrightarrow X \times X$ are open ([JT84]).

The category of locales has finite products, indeed all limits. This is because the theory of frames is suitably algebraic and so colimits can be described using generators and relations in the usual manner. An explicit description of a frame constructed from its generators and relations is given by Johnstone in II 2.11 of [J82]. Frame coproduct is given by suplattice tensor, [JT84]; for a locale X the diagonal map is given by

$$\begin{aligned} \Omega\Delta_X : \Omega X \otimes \Omega X &\longrightarrow \Omega X \\ a \otimes b &\longmapsto a \wedge b. \end{aligned}$$

Broadly, locales behave well under change of base. If $f : \mathcal{F} \to \mathcal{E}$ is a geometric morphism between elementary toposes then f_* preserves the property of being a frame (and of being a frame homomorphism). The direct image

[1] Note that we use the term *set* to mean an object of a topos.

functor of f, $f_* : \mathcal{F} \to \mathcal{E}$, therefore gives rise to a functor, denoted $\Sigma_f : \mathbf{Loc}_\mathcal{F} \to \mathbf{Loc}_\mathcal{E}$[2]. If we define $f^* : \mathbf{Loc}_\mathcal{E} \to \mathbf{Loc}_\mathcal{F}$ by

$$\Omega_\mathcal{F} f^* X \cong \mathbf{Fr}_\mathcal{F}\langle f^* G_X \mid f^* R_X \rangle$$

where G_X and R_X are generators and relations for an arbitrary frame $\Omega_\mathcal{E} X$ of $\mathcal{E}$, then f^* is right adjoint to Σ_f ([JT84]). It is known, for example C2.4 of [J02], that in this situation

$$\begin{array}{ccc} Sh_\mathcal{F}(f^*X) & \xrightarrow{f_X} & Sh_\mathcal{E}(X) \\ \downarrow \gamma^{f^*X} & & \downarrow \gamma^X \\ \mathcal{F} & \xrightarrow{f} & \mathcal{E} \end{array}$$

is a pullback diagram in the category of toposes. The term *pullback functor* is therefore applied to f^*. Since f_X is a geometric morphism it too induces an adjunction $\Sigma_{f_X} \dashv f_X^* : \mathbf{Loc}_\mathcal{F}/f^*X \rightleftarrows \mathbf{Loc}_\mathcal{E}/X$ for which we will need the following explicit description:

Lemma 3.1 *For any geometric morphism $f : \mathcal{F} \to \mathcal{E}$*
(i) $f_X^*(W \xrightarrow{g} X) \cong f^*W \xrightarrow{f^*g} f^*X$ and
(ii) $\Sigma_{f_X}(Y \xrightarrow{h} f^*X) \cong$ the adjoint transpose of h.

Proof. We only need to prove (i) since (ii) then follows by uniqueness of adjoints. But Theorem C2.4.11 of [J02] proves that Beck-Chevalley holds for the diagram of adjunctions

$$\begin{array}{ccc} \mathbf{Loc}_\mathcal{F}/f^*X & \rightleftarrows & \mathbf{Loc}_\mathcal{E}/X \\ \uparrow\downarrow & & \uparrow\downarrow \\ \mathbf{Loc}_\mathcal{F} & \rightleftarrows & \mathbf{Loc}_\mathcal{E} \end{array}$$

and this forces (i) so we are done. ■

Note that the adjunction $\Sigma_f \dashv f^*$ is *order enriched*, meaning that the homset natural bijections

$$\mathbf{Loc}_\mathcal{E}(\Sigma_f W, X) \cong \mathbf{Loc}_\mathcal{F}(W, f^*X)$$

are order isomorphisms.

For any locales X and Y, dcpo homomorphisms from ΩX to ΩY (i.e. directed join preserving maps) are in order isomorphism with

$$Nat[\mathbf{Loc}(\_ \times X, \mathbb{S}), \mathbf{Loc}(\_ \times Y, \mathbb{S})]$$

where $\mathbf{Loc}(\_ \times X, \mathbb{S}) : \mathbf{Loc}^{op} \to \mathbf{Set}$ is the presheaf for any locale X and $Nat[\_]$ is the collection of natural transformations ordered componentwise in the obvious manner. Of course $\mathbb{S}$ is the *Sierpiński locale*, i.e. that locale whose frame of

[2]The Σ_f notation is consistent with its usual usage as the 'post compose with f' functor when f is a geometric morphism between localic toposes since in such a situation $\mathbf{Loc}_\mathcal{F}$ and $\mathbf{Loc}_\mathcal{E}$ are both slices via Equation 1.

opens is the free frame on the singleton set subject to no relations. It is an *order* internal meet semilattice in **Loc** meaning that the meet operation is right adjoint to the diagonal. Note that by construction f^* will always preserve the Sierpiński locale.

This isomorphism is an extension, to dcpo homomorphisms, of the mapping:

$$\Omega f \longmapsto \mathbf{Loc}(\_ \times f, \mathbb{S})$$

for any frame homomorphism $\Omega f : \Omega Y \to \Omega X$. Since it can be verified using Yoneda's lemma that $\mathbf{Loc}(\_ \times Y, \mathbb{S})$ is the exponential $\mathbf{Loc}(\_, \mathbb{S})^{\mathbf{Loc}(\_,Y)}$ the notation $\mathbb{S}^Y$ is used for the presheaf $\mathbf{Loc}(\_ \times Y, \mathbb{S})$; it is not generally an object of **Loc**.

The result on the representation of dcpo homomorphisms as natural transformations is originally in [VT04], though the construction in [T04] is the one that we are using in this paper since we need to call on:

Lemma 3.2 *For any locale X there is an order isomorphism between the poset of monotone maps $B \to \Omega X$ and*

$$\mathbf{Loc}(Idl(B) \times X, \mathbb{S})$$

for any poset B. This order isomorphism is natural in B and in dcpo homomorphism between ΩX.

Here $Idl(B)$, the *ideal completeion* of B, is that locale whose frame of opens is $\mathcal{U}B$, the set of upper closed subsets of B. It is called the ideal completion since its point are in order isomorphism with the ideals (lower closed, directed) subsets of B. Note that $\mathcal{U}B$ is the splitting of the idempotent $\uparrow : PB \to PB$.

Proof. This result, without the naturality statement, is well known lattice theory. For the naturality component consult [T04] (Lemmas 51 and 54 together with the remarks after Definition 52). ■

4 $\Sigma_f \dashv f^*$ satisfies Frobenius reciprocity

Our first task is to show that the adjunction $\Sigma_f \dashv f^*$ satisfies Frobenius reciprocity for any geometric morphism $f : \mathcal{F} \to \mathcal{E}$. Let us recall (e.g. A.1.5.8 of [J02]) the definition of this condition on an adjunction.

Definition 4.1 *An adjunction $L \dashv R : \mathcal{D} \rightleftarrows \mathcal{C}$ between cartesian categories satisfies* Frobenius reciprocity *provided the map $n_{X,W} : L(R(X) \times W) \overset{(L\pi_1, L\pi_2)}{\longrightarrow} LRX \times LW \overset{\varepsilon_X \times Id_{LW}}{\longrightarrow} X \times LW$ is an isomorphism for all objects W and X of $\mathcal{D}$ and $\mathcal{C}$ respectively.*

Theorem 4.2 *Given a geometric morphism $f : \mathcal{F} \to \mathcal{E}$, the induced order enriched adjunction $\Sigma_f \dashv f^* : \mathbf{Loc}_{\mathcal{F}} \rightleftarrows \mathbf{Loc}_{\mathcal{E}}$ satisfies Frobenius reciprocity.*

Proof. This is again an application of Theorem C2.4.11 of [J02] since

$$\begin{array}{ccc} Sh_{\mathcal{F}}(f^*X) & \stackrel{\gamma^{f^*X}}{\longrightarrow} & \mathcal{F} \\ \downarrow f_X & & \downarrow f \\ Sh_{\mathcal{E}}(X) & \stackrel{\gamma^X}{\longrightarrow} & \mathcal{E} \end{array}$$

is also a pullback diagram of toposes. Therefore Beck-Chevalley also holds for the square

$$\begin{array}{ccc} \mathbf{Loc}_{\mathcal{F}}/f^*X & \rightleftarrows & \mathbf{Loc}_{\mathcal{F}} \\ \uparrow\downarrow & & \uparrow\downarrow \\ \mathbf{Loc}_{\mathcal{E}}/X & \rightleftarrows & \mathbf{Loc}_{\mathcal{E}} \end{array}$$

of adjunctions.

The Beck-Chevalley condition, evaluated at a locale W over $\mathcal{F}$, implies that $\phi_W : \Sigma_{f_X}(\gamma^{f^*X})^*W \to (\gamma^X)^*\Sigma_f W$ is an isomorphism where ϕ_W is the adjoint transpose, via $\Sigma_{f_X} \dashv f_X^*$, of

$$(\gamma^{f^*X})^*W \stackrel{(\gamma^{f^*X})^*(\eta_W)}{\longrightarrow} (\gamma^{f^*X})^*f^*\Sigma_f W \stackrel{\cong}{\to} f_X^*(\gamma^X)^*\Sigma_f W$$

and η is the unit of the adjunction. However this morphism is

$$f^*X \times W \stackrel{Id_{f^*X}\times\eta_W}{\longrightarrow} f^*X \times f^*\Sigma_f W \stackrel{(f^*\pi_1, f^*\pi_2)^{-1}}{\longrightarrow} f^*(X \times \Sigma_f W)$$

and since it can be checked that this is the adjoint transpose of $n_{W,X}$ we are done. ∎

5 Properties of adjunctions between categories of locales

We now exploit an assumption of Frobenius reciprocity to extend any order enriched adjunction $L \dashv R : \mathbf{Loc}_{\mathcal{F}} \rightleftarrows \mathbf{Loc}_{\mathcal{E}}$ to dcpo homomorphisms. This extension is available since dcpo homomorphisms between frames can be represented as natural transformations. Let $\overline{\mathbf{Loc}}^{op}$ denote the category whose objects are frames and whose morphisms are dcpo homomorphisms. It is equivalent to the full subcategory of $[\mathbf{Loc}^{op}, \mathbf{Set}]$ whose objects are the presheaves $\mathbb{S}^X$. (In fact this category is equivalent to the opposite of the Kleisli category of the double power monad, see [JV91], [V04] and [VT04] for background; however we shall not use the double power locale construction in this paper.)

Proposition 5.1 *For any order enriched adjunction $L \dashv R : \mathbf{Loc}_{\mathcal{F}} \rightleftarrows \mathbf{Loc}_{\mathcal{E}}$ which satisfies Frobenius reciprocity and for which there exists an isomorphism $R\mathbb{S}_{\mathcal{E}} \cong \mathbb{S}_{\mathcal{F}}$, there exists an order enriched adjunction $\overline{R}^{op} \dashv \overline{L}^{op} : \overline{\mathbf{Loc}_{\mathcal{F}}}^{op} \rightleftarrows \overline{\mathbf{Loc}_{\mathcal{E}}}^{op}$ such that the squares*

$$\begin{array}{ccc} \mathbf{Loc}_{\mathcal{F}} & \stackrel{\mathbb{S}_{\mathcal{F}}^{(\text{-})}}{\hookrightarrow} & \overline{\mathbf{Loc}_{\mathcal{F}}}^{op} \\ L \downarrow\dashv\uparrow R & & \overline{L}^{op} \downarrow\vdash\uparrow \overline{R}^{op} \\ \mathbf{Loc}_{\mathcal{E}} & \stackrel{\mathbb{S}_{\mathcal{E}}^{(\text{-})}}{\hookrightarrow} & \overline{\mathbf{Loc}_{\mathcal{E}}}^{op} \end{array}$$

commute; i.e. $\overline{R}^{op} \dashv \overline{L}^{op}$ *extends* $L \dashv R$ *contravariantly.*

Proof. We must show how to define $\overline{R}^{op}$ and $\overline{L}^{op}$ on natural transformations. Say $\alpha : \mathbb{S}_{\mathcal{E}}^X \to \mathbb{S}_{\mathcal{E}}^{X'}$ define $\overline{R}^{op}(\alpha)$ by

$$\begin{array}{ccc}
\mathbf{Loc}_{\mathcal{F}}(W \times RX, \mathbb{S}_{\mathcal{F}}) & \stackrel{[\overline{R}^{op}(\alpha)]_W}{\longrightarrow} & \mathbf{Loc}_{\mathcal{F}}(W \times RX', \mathbb{S}_{\mathcal{F}}) \\
\downarrow\cong & & \uparrow\cong \\
\mathbf{Loc}_{\mathcal{F}}(W \times RX, R\mathbb{S}_{\mathcal{E}}) & & \mathbf{Loc}_{\mathcal{F}}(W \times RX', R\mathbb{S}_{\mathcal{E}}) \\
\downarrow\cong & & \uparrow\cong \\
\mathbf{Loc}_{\mathcal{E}}(LW \times X, \mathbb{S}_{\mathcal{E}}) & \stackrel{\alpha_{LW}}{\longrightarrow} & \mathbf{Loc}_{\mathcal{E}}(LW \times X', \mathbb{S}_{\mathcal{E}})
\end{array}$$

for any locale W of $\mathcal{F}$. Certainly the vertical morphisms are isomorphisms by assumption. Similarly define $\overline{L}^{op}$ by sending any $\beta : \mathbb{S}_{\mathcal{F}}^W \to \mathbb{S}_{\mathcal{F}}^{W'}$ to $\overline{L}^{op}(\beta)$ defined by

$$\begin{array}{ccc}
\mathbf{Loc}_{\mathcal{E}}(X \times LW, \mathbb{S}_{\mathcal{E}}) & \stackrel{[\overline{L}^{op}(\beta)]_X}{\longrightarrow} & \mathbf{Loc}_{\mathcal{E}}(X \times LW', \mathbb{S}_{\mathcal{E}}) \\
\downarrow\cong & & \uparrow\cong \\
\mathbf{Loc}_{\mathcal{F}}(RX \times W, R\mathbb{S}_{\mathcal{E}}) & & \mathbf{Loc}_{\mathcal{F}}(RX \times W', R\mathbb{S}_{\mathcal{E}}) \\
\downarrow\cong & & \uparrow\cong \\
\mathbf{Loc}_{\mathcal{F}}(RX \times W, \mathbb{S}_{\mathcal{F}}) & \stackrel{\beta_{RX}}{\longrightarrow} & \mathbf{Loc}_{\mathcal{F}}(RX \times W', \mathbb{S}_{\mathcal{F}})
\end{array}$$

That $\overline{R}^{op}$ is left adjoint to $\overline{L}^{op}$ follows by applying $\mathbb{S}^{(\text{-})}$ to the triangular identities of the adjunction $L \dashv R$. It is clear that the functors $\overline{R}^{op}$ and $\overline{L}^{op}$ are order enriched and so too, therefore, is the adjunction. ∎

A key application of this extension is that with it R must preserve certain properties which we now establish.

Proposition 5.2 *R preserves the property of being a discrete locale and so defines a functor $R : \mathcal{E} \to \mathcal{F}$.*

Proof. Given that R, as a right adjoint, preserves limits it is sufficient to check that R preserves open maps as we have outlined above how the property of being a discrete locale can by characterized in terms of having open finite diagonals. Say $f : X \to Y$ is an open map, then there is $\exists_f : \Omega_{\mathcal{E}} X \to \Omega_{\mathcal{E}} Y$ left adjoint to $\Omega_{\mathcal{E}} f$ and satisfying

$$\begin{array}{ccc}
\Omega_{\mathcal{E}} X \otimes \Omega_{\mathcal{E}} Y & \stackrel{\exists_f \otimes 1}{\longrightarrow} & \Omega_{\mathcal{E}} Y \otimes \Omega_{\mathcal{E}} Y \\
1 \otimes \Omega_{\mathcal{E}} f \downarrow & & \\
\Omega_{\mathcal{E}} X \otimes \Omega_{\mathcal{E}} X & & \downarrow \Omega \Delta_Y \\
\Omega \Delta_X \downarrow & & \\
\Omega_{\mathcal{E}} X & \stackrel{\exists_f}{\longrightarrow} & \Omega_{\mathcal{E}} Y
\end{array}$$

$\exists_f$, as a left adjoint, is certainly a dcpo homomorphism and so $\overline{R}^{op}(\exists_f)$ is well defined and is a left adjoint to $\overline{R}^{op}(\Omega_{\mathcal{E}} f)$ as it is stipulated that $L \dashv R$ is order enriched and so $\overline{R}^{op}$ is order enriched. Noting that $\overline{R}^{op}(\exists_f) \otimes 1 = \overline{R}^{op}(\exists_f \otimes 1)$ as both are left adjoint to

$$\begin{aligned} \overline{R}^{op}(\Omega_{\mathcal{E}} f \otimes 1) &= \overline{R}^{op}(\Omega_{\mathcal{E}}(f \times 1)) \\ &= \Omega_{\mathcal{F}}(R(f \times 1)) \\ &= \Omega_{\mathcal{F}}(Rf \times R1)) \\ &= \overline{R}^{op}(\Omega_{\mathcal{E}} f) \otimes 1, \end{aligned}$$

we have, by applying $\overline{R}^{op}$ to the commuting diagram, that Rf is open and we are done. ■

We now need a technical lemma which will ease the proof of the next property of R.

Lemma 5.3 *If $h : X' \to X$ is a locale map in $\mathcal{E}$ and $b : Y \to \mathbb{S}_{\mathcal{E}}$ an open of Y, another locale over $\mathcal{E}$, then consider the natural transformation $\mathbb{S}_{\mathcal{E}}^{h\pi_1}(\_) \sqcap_{\mathbb{S}_{\mathcal{E}}} b : \mathbb{S}_{\mathcal{E}}^{X} \to \mathbb{S}_{\mathcal{E}}^{X' \times Y}$ defined by*

$$\begin{aligned} \mathbf{Loc}_{\mathcal{E}}(Z \times X, \mathbb{S}_{\mathcal{E}}) &\to \mathbf{Loc}_{\mathcal{E}}(Z \times X' \times Y, \mathbb{S}_{\mathcal{E}}) \\ I &\mapsto \sqcap_{\mathbb{S}_{\mathcal{E}}}(I(Id_Z \times h)\pi_{12}, b\pi_3) \end{aligned}$$

for each Z, where $\sqcap_{\mathbb{S}_{\mathcal{E}}}$ is the meet operation of the Sierpiński locale. Then $\overline{R}^{op}$ preserves $\mathbb{S}_{\mathcal{E}}^{h\pi_1}(\_) \sqcap_{\mathbb{S}_{\mathcal{E}}} b$; i.e.

$$\overline{R}^{op}(\mathbb{S}_{\mathcal{E}}^{h\pi_1}(\_) \sqcap_{\mathbb{S}_{\mathcal{E}}} b) \quad \cong \quad \mathbb{S}_{\mathcal{F}}^{(Rh)\pi_1}(\_) \sqcap_{\mathbb{S}_{\mathcal{F}}} Rb.$$

Proof. If W is a locale over $\mathcal{F}$ then $[\overline{R}^{op}(\mathbb{S}_{\mathcal{E}}^{h\pi_1}(\_) \sqcap_{\mathbb{S}_{\mathcal{E}}} b)]_W(I')$, for $I' : W \times RX \to R\mathbb{S}_{\mathcal{E}}$, is equal to $\widetilde{Jn_{W,X' \times Y}}(Id_W \times (R\pi_1, R\pi_2)^{-1})$ where J is

$$\sqcap_{\mathbb{S}_{\mathcal{E}}}(\tilde{I'} n_{W,X}^{-1}(Id_{LW} \times h)\pi_{12}, b\pi_3)$$

and $\tilde{(\_)}$ denotes taking adjoint transpose via $L \dashv R$. We are passing through the isomorphism $R\mathbb{S}_{\mathcal{F}} \cong \mathbb{S}_{\mathcal{E}}$ without notation. Checking that this gives $\sqcap_{\mathbb{S}_{\mathcal{F}}}(I'(Id_W \times Rh)\pi_{12}, Rb\pi_3)$ as required is a routine diagram chase. Note that $\sqcap_{\mathbb{S}_{\mathcal{F}}} \cong R(\sqcap_{\mathbb{S}_{\mathcal{E}}})$ since both are right adjoint to the diagonal. ■

Our final property of R that can be established because of its extension to $\overline{R}^{op}$, is that R commutes with the ideal completion construction $Idl(\_)$.

Proposition 5.4 *If $(B, \leq_B)$ is a poset of $\mathcal{E}$ then $R(Idl_{\mathcal{E}}(B)) \cong Idl_{\mathcal{F}}(RB)$.*

Proof. $\Omega_{\mathcal{E}} Idl(B)$ is the splitting of the idempotent $\uparrow_B : PB \to PB$. This is certainly a dcpo homomorphism and so we have but to check that $\overline{R}^{op}(\uparrow_B) = \uparrow_{RB}$. However the natural transformation corresponding to $\uparrow_B$ is given by

$$\mathbb{S}_{\mathcal{E}}^{B} \overset{\mathbb{S}_{\mathcal{E}}^{\pi_1}(\_) \sqcap_{\mathbb{S}_{\mathcal{E}}} \leq_B}{\longrightarrow} \mathbb{S}_{\mathcal{E}}^{B \times B \times B \times B} \overset{\mathbb{S}_{\mathcal{E}}^{\Delta_{B \times B}}}{\longrightarrow} \mathbb{S}_{\mathcal{E}}^{B \times B} \overset{\exists_{\pi_2}}{\to} \mathbb{S}_{\mathcal{E}}^{B}.$$

The first term in the composition is preserved by $\overline{R}^{op}$ using the lemma (take $\pi_1 : B \times B \to B$ as h, $\leq_B = b$ and $B \times B$ as Y). The second term is preserved since R is a right adjoint and so preserves the diagonal. The third term is preserved by $\overline{R}^{op}$ since we have checked that this extension is order enriched and so preserves the property of being a left adjoint. ■

6 The representation theorem

Discrete locales are closed under finite limits and so the restricted functor $R : \mathcal{E} \to \mathcal{F}$ is cartesian: it is our candidate for the inverse image of a geometric morphism. Finding its right adjoint, and so proving that it is an inverse image of a geometric morphism, exploits the following lemma:

Lemma 6.1 *Any object A in a topos exists as an equalizer*

$$A \hookrightarrow \Omega X_A \underset{h_A}{\overset{g_A}{\rightrightarrows}} \Omega Y_A$$

of frames where g_A and h_A are dcpo homomorphisms.

Proof. Take $\Omega X_A = PA$, the power set of A, and take $\Omega Y_A = P(A \times A) \times \Omega$ (which is the frame of the locale $(A \times A) + 1$). Recall $\Omega = P1$, and $1 = \{*\}$ the singleton set. Let $g_A(I) = (I \times I, \{*\})$ and let $h_A(I) = (\{(i,i) \mid i \in I\}, \{* \mid \exists i \in I\})$. It is routine to verify that these are both dcpo homomorphisms and that A is their equalizer. ■

It therefore makes sense to define $R_* : \mathcal{F} \to \mathcal{E}$ by sending an object A of $\mathcal{F}$ to the equalizer in $\mathcal{E}$ of the diagram

$$\overline{L}^{op}\Omega_{\mathcal{F}} X_A \underset{\overline{L}^{op} h_A}{\overset{\overline{L}^{op} g_A}{\rightrightarrows}} \overline{L}^{op}\Omega_{\mathcal{F}} Y_A$$

where the $\Omega_{\mathcal{F}} X_A$, g_A etc. are constructed as in the lemma (carried out in the topos $\mathcal{F}$).

Proposition 6.2 *R_* is right adjoint to R.*

Proof. We need to show that morphisms $\overline{\phi} : B \to R_* A$ of $\mathcal{E}$ are in natural bijection with morphisms $RB \to A$ of $\mathcal{F}$. By construction of $R_* A$ such $\overline{\phi}$ are in bijection with morphisms $\phi : B \to \overline{L}^{op}\Omega_{\mathcal{F}} X_A$ such that $\overline{L}^{op} g_A \phi = \overline{L}^{op} h_A \phi$. But morphisms $B \to \overline{L}^{op}\Omega_{\mathcal{F}} X_A$ are in bijection with $\mathbf{Loc}_{\mathcal{E}}(L(X_A) \times B, \mathbb{S}_{\mathcal{E}})$ by applying Lemma 3.2 (treating B as a poset with a discrete ordering). Then using the naturality of that lemma and noting the commutative squares

$$\begin{array}{ccc} \mathbf{Loc}_{\mathcal{E}}(L(X_A) \times B, \mathbb{S}_{\mathcal{E}}) & \begin{array}{c} (\overline{L}^{op} g_A)_B \\ \rightarrow \\ \rightarrow \\ (\overline{L}^{op} h_A)_B \end{array} & \mathbf{Loc}_{\mathcal{E}}(L(Y_A) \times B, \mathbb{S}_{\mathcal{E}}) \\ \cong \ \downarrow & & \downarrow \ \cong \\ \mathbf{Loc}_{\mathcal{F}}(X_A \times R(B), \mathbb{S}_{\mathcal{F}}) & \begin{array}{c} (g_A)_{R(B)} \\ \rightarrow \\ \rightarrow \\ (h_A)_{R(B)} \end{array} & \mathbf{Loc}_{\mathcal{F}}(Y_A \times R(B), \mathbb{S}_{\mathcal{F}}) \end{array}$$

we obtain a bijection with elements of $\mathbf{Loc}_{\mathcal{F}}(X_A \times R(B), \mathbb{S}_{\mathcal{F}})$ and hence to morphisms $R(B) \to \Omega_{\mathcal{E}} X_A$ that compose equally with g_A and h_A. ■

We now state and prove the main result.

Theorem 6.3 *For any two elementary toposes $\mathcal{F}$ and $\mathcal{E}$ there is an equivalence between the category of geometric morphisms $\mathcal{F}$ to $\mathcal{E}$ and the category of order enriched adjunctions $L \dashv R$ between $\mathbf{Loc}_{\mathcal{F}}$ and $\mathbf{Loc}_{\mathcal{E}}$ which satisfy Frobenius reciprocity and for which $R\mathbb{S}_{\mathcal{E}} \cong \mathbb{S}_{\mathcal{F}}$.*

Proof. We have just shown how to construct a geometric morphism from an adjunction satisfying Frobenius reciprocity and for which $R\mathbb{S}_{\mathcal{E}} \cong \mathbb{S}_{\mathcal{F}}$. It is clear from construction that if the adjunction between locales is of the form $\Sigma_f \dashv f^*$ for a geometric morphism f then f is recovered. This is because the pullback functor and the inverse image functor are isomorphic both viewed as actions on discrete locales.

In the other direction say we are given an adjunction $L \dashv R$. The corresponding geometric morphism has been constructed: $R \dashv R_*$. This geometric morphism gives rise to an adjunction satisfying Frobenius reciprocity, its left adjoint, denoted $L' : \mathbf{Loc}_{\mathcal{F}} \to \mathbf{Loc}_{\mathcal{E}}$ say, being defined by setting

$$\Omega_{\mathcal{E}} L'(W) \equiv R_* \Omega_{\mathcal{F}} W$$

for any locale W of $\mathcal{F}$. But for any poset B of $\mathcal{E}$, using the notation $\mathbf{Pos}_{\mathcal{E}}$ for the category of posets over $\mathcal{E}$, we have

$$\begin{array}{rcl} \mathbf{Pos}_{\mathcal{E}}(B, R_* \Omega_{\mathcal{F}} W) & \cong & \mathbf{Pos}_{\mathcal{F}}(RB, \Omega_{\mathcal{F}} W) \\ & \cong & \mathbf{Loc}_{\mathcal{F}}(W \times Idl_{\mathcal{F}}(RB), \mathbb{S}_{\mathcal{F}}) \\ & \cong & \mathbf{Loc}_{\mathcal{F}}(W \times RIdl_{\mathcal{E}} B, \mathbb{S}_{\mathcal{F}}) \\ & \cong & \mathbf{Loc}_{\mathcal{E}}(L(W) \times Idl_{\mathcal{E}} B, \mathbb{S}_{\mathcal{E}}) \\ & \cong & \mathbf{Pos}_{\mathcal{E}}(B, \Omega_{\mathcal{E}} L(W)) \end{array}$$

and so $L \cong L'$ using the naturality, in B, of Lemma 3.2. ■

7 Conclusions

The intended application of this result is a new categorical account of the theory of geometric morphisms. If the category of locales can be axiomatised (see, for example, [T05]) then it is clear from the representation theorem just offered what the definition of morphism between such categories should be, i.e. order enriched adjunctions that preserve the (suitably axiomatised) Sierpiński object and satisfying Frobenius reciprocity. If such an account is viable then it would have an advantage over the current view of geometric morphisms, the advantage being that open and proper maps would be dual concepts (the duality is obtained by reversing the order enrichment). This would represent a step towards formally showing that the theory of proper geometric morphisms and the theory of open geometric morphisms are, in fact, dual aspects of the same theory thus formalising categorically a relationship that has been, so far, only intuitively understood.

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