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A category of 'spaces' that is not a category of Locales.

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Outline: Properties Loc \longrightarrow Axioms on \mathcal{C} $\longrightarrow \exists \mathcal{C} \neq \underline{\text{Loc}}_{\mathcal{E}} \forall \text{topos } \mathcal{E}.$

⋮

Properties of Loc

1. Loc order enriched, finite products.

2. $\exists \mathcal{F}$, internal distributive lattice, s.t. $\forall X$ $\mathcal{F}^{\mathcal{F}^X}$ exists.

Where \mathcal{F}^X is the presheaf

$$\underline{\text{Loc}}^{\text{op}} \longrightarrow \underline{\text{Set}}$$

$$Y \longmapsto \underline{\text{Loc}}(Y \times X, \mathcal{F})$$

$$\text{IPX} \equiv \mathcal{F}^{\mathcal{F}^X}; \forall \text{ locales } Y$$

$$\underline{\text{Loc}}(Y, \text{IPX}) \cong \underline{\text{Nat}}[\mathcal{F}^X, \mathcal{F}^Y]$$

If IPX so defined, it is a strong monad

i.e. $(y \mathcal{F})^{(\mathcal{F}^X)}$ exists in $[\underline{\text{Loc}}^{\text{op}}, \underline{\text{Set}}]$ and is representable. ($y = \text{Yoneda}$)

Definition: \mathcal{C} is a category of spaces if 1. & 2.

Example: $\underline{\text{Loc}}_{\mathcal{E}}$ for any elementary topos \mathcal{E} .

For any $f: X \rightarrow Y$ in a category of spaces f is open if $\exists \exists_f: \mathcal{P}^X \rightarrow \mathcal{P}^Y$ s.t. $\exists_f^{-1} \mathcal{P}^f$ and $\exists_f(a \cap \mathcal{P}^f b) = \exists_f(a) \cap \mathcal{P}^f b$.

X is discrete if $X \xrightarrow{!} 1$ and $\Delta_X: X \rightarrow X \times X$ are open; $\text{DIS}_{\mathcal{C}} = \text{full-subcategory}$

For $\underline{\text{Loc}}$ usual notions are recovered.

Recall, \forall topos \mathcal{E} ,

$$\mathcal{E} \cong \text{DIS}_{\underline{\text{Loc}}_{\mathcal{E}}} \hookrightarrow \underline{\text{Loc}}_{\mathcal{E}}.$$

'Nice' general categorical result about double exponentiation:

Proposition: If \mathcal{C} is a category with finite products and G an internal group. Given an object A s.t. A^{A^X} exists $\forall X$, then $(A, \pi_2)^{(A, \pi_2)}^{(X, a)}$ exists in $[\mathcal{C}, \mathcal{C}]$ for each G -object (X, a) .

\nearrow
 G -objects & G -homomorphisms.
e.g. $(X, a: G \times X \rightarrow X)$

Proof We show $(A, \pi_2)^{(A, \pi_2)^{(X, a)}} = (IPX, \zeta \times IPX \xrightarrow{t_{\zeta \times X}} IP(\zeta \times X) \xrightarrow{IPa} IPX)$

(3)

$t =$ strength on IP .

Need to show $\forall (Y, b)$

$$[\zeta, \mathcal{C}](Y, b), (IPX, IPa t_{\zeta, X}) \cong \underline{\text{Nat}} \left[(A, \pi_2)^{(X, a)}, (A, \pi_2)^{(Y, b)} \right]$$

$$\underline{\text{LHS}} \cong \left\{ Y \xrightarrow{\phi} IPX \mid \phi \text{ a } \zeta\text{-hom.} \right\} \cong \left\{ \kappa: A^X \rightarrow A^Y \mid \begin{array}{ccc} A^X & \xrightarrow{\kappa} & A^Y \\ A^a \downarrow & & \downarrow A^b \\ A^{\zeta \times X} & \xrightarrow{\alpha_{\zeta}} & A^{\zeta \times Y} \end{array} \right\}$$

exp. transpose

But

$$(A, \pi_2)^{(Y, b)} \xrightarrow{A^b} (A, \pi_2)^{(\zeta \times Y, m \times \text{id}_Y)} \xrightarrow[A \text{ Id} \times b]{A^{m \times \text{id}_Y}} (A, \pi_2)^{(\zeta \times \zeta \times Y, m \times \text{id}_{\zeta \times Y})}$$

an equalizer (apply $(A, \pi_2)^{(-)}$ to U -split coequalizer $\zeta \times \zeta \times Y \xrightarrow[\text{Id}_{\zeta} \times b]{m \times \text{id}_Y} \zeta \times Y \xrightarrow{b} Y$)

$$\Rightarrow \underline{\text{RHS}} \hookrightarrow \underline{\text{Nat}} \left[(A, \pi_2)^{(X, a)}, (A, \pi_2)^{(\zeta \times Y, m \times \text{id}_Y)} \right] \cong \underline{\text{Nat}} \left[A^{U(X, a)}, A^Y \right] = \underline{\text{Nat}} \left[A^X, A^Y \right] \quad \square$$

$$\mathcal{C} \xrightleftharpoons[u]{\zeta \times (-)} [\zeta, \mathcal{C}] \text{ sat. Frobenius rec. } \Rightarrow \text{extends } A^{U(-)} \dashv (A, \pi_2)^{\zeta \times (-)}$$

Certainly: \mathcal{C} order enriched $\Rightarrow [G, \mathcal{C}]$ order enriched
 \mathcal{C} finite products $\Rightarrow [G, \mathcal{C}]$ has finite products
 $\exists \text{ \& } \text{dlat in } \mathcal{C} \Rightarrow (\$, \pi_2) \text{ DLAT in } [G, \mathcal{C}]$
 $\vdots \Rightarrow \vdots$

category of spaces

can extend G -stability to axioms, so that \mathcal{C} captures a number of key results from locale theory e.g. open/proper stability, Hof. Mislove...

So by Prop. double exponentiability also G -stable:

Theorem \mathcal{C} a category of spaces. Then so is $[G, \mathcal{C}]$ for any internal group G . \square

Define $BG =$ full subcategory of $[G, \mathcal{C}]$ consisting of (X, a) s.t. X discrete.

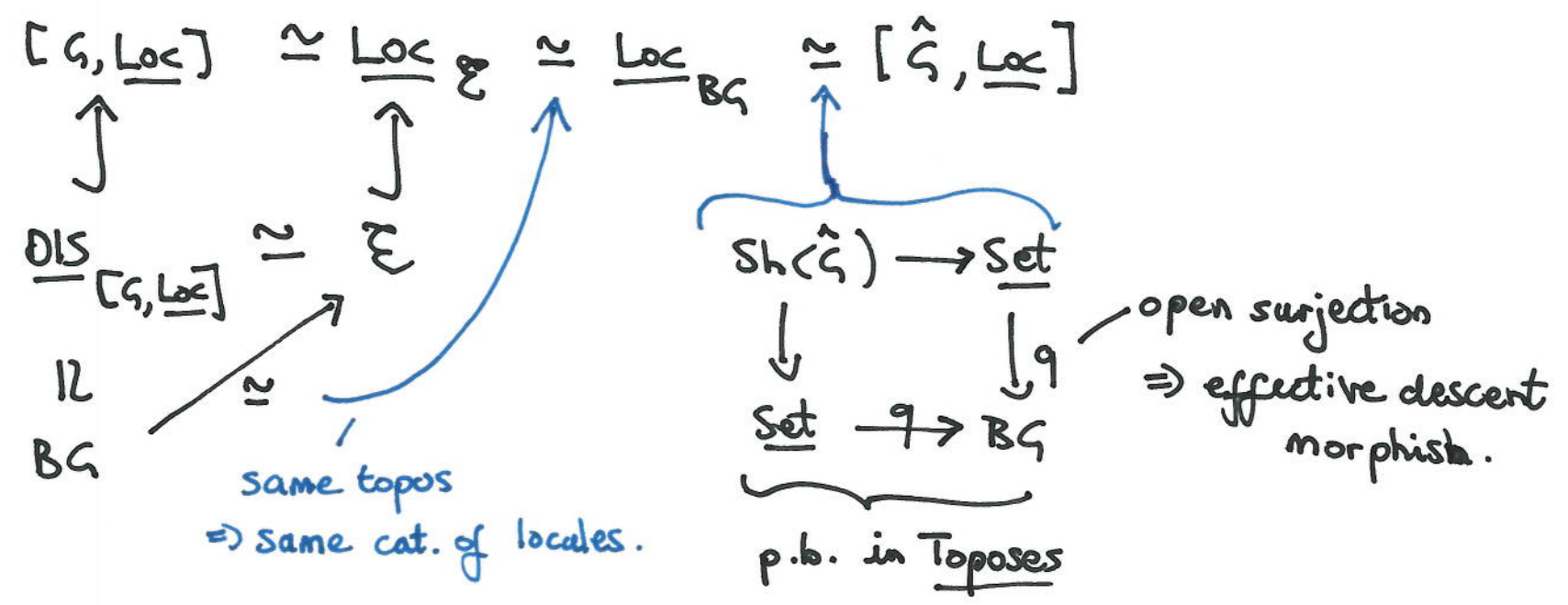
Well known (or show axiomatically...) \forall localic groups: $DIS_{[G, \underline{Loc}]} \cong BG$.

But:

Prop. There exist category of spaces that are not categories of locales.

Proof \mathcal{G} an open localic group. Say $[\mathcal{G}, \underline{\text{Loc}}] \cong \underline{\text{Loc}}_{\mathcal{E}}$ some elementary topos \mathcal{E} .

Then,



This \cong (between $[\mathcal{G}, \underline{\text{Loc}}], [\hat{\mathcal{G}}, \underline{\text{Loc}}]$) is under $\underline{\text{Loc}}$ (i.e. commutes with $u \mapsto \mathcal{G} \times (-), u \mapsto \hat{\mathcal{G}} \times (-)$) and so $\mathcal{G} \cong \hat{\mathcal{G}}$ (i.e. \mathcal{G} étale complete). Take any non-étale complete to finish. \square

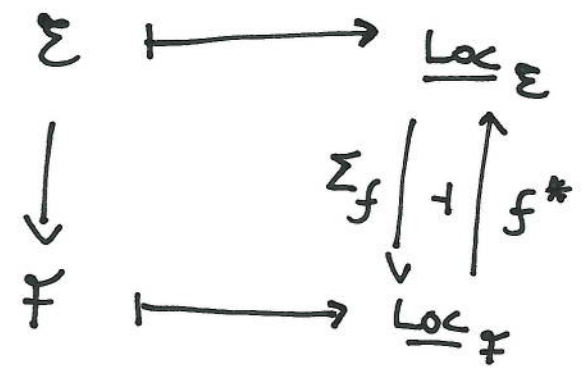


Defn \mathcal{R} by $ob(\mathcal{R}) = \text{categories of spaces.}$

$$\mathcal{M}(\mathcal{R}) = \{ L \dashv R : \mathcal{D} \rightleftarrows \mathcal{C} \mid R \text{ preserves } +, \exists \text{ iso. } R \mathbb{P}_{\mathcal{C}} \cong \mathbb{P}_{\mathcal{D}} R \}$$

Then \exists Toposes $\longrightarrow \mathcal{R}$

full and faithful: not essentially surjective.



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