A category of 'spaces' that is not a category of Locales.

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Outline: Propeties $\underline{Loc}$ $\longrightarrow$ Axioms on $\mathcal{C}$ $\longrightarrow$ $\exists\, \mathcal{C} \not\simeq \underline{Loc}_{\mathcal{E}}$ $\forall$ topos $\mathcal{E}$.

Properties of $\underline{Loc}$

1. $\underline{Loc}$ order enriched, finite products.

2. $\exists\, \$$, internal distributive lattice, s.t. $\forall X$ $\$^{\$^X}$ exists.

   i.e. $(y\,\$)^{(\$^X)}$ exists in $[\underline{Loc}^{op}, \underline{Set}]$ and is representable. (y = Yoneda)

   Where $\$^X$ is the presheaf

$$\underline{Loc}^{op} \longrightarrow \underline{Set}$$
$$Y \longmapsto \underline{Loc}(Y \times X, \$)$$

$\mathbb{P}X \equiv \$^{\$^X}$; $\forall$ locales Y

$$\underline{Loc}(Y, \mathbb{P}X) \cong \underline{Nat}[\$^X, \$^Y]$$

If $\mathbb{P}X$ so defined, it is a strong monad

Definition: $\mathcal{C}$ is a <u>category of spaces</u> if 1. & 2.

Example: $\underline{\mathrm{Loc}}_{\mathcal{E}}$ for any elementary topos $\mathcal{E}$.

For any $f: X \to Y$ in a category of spaces f is <u>open</u> if $\exists$ $\exists_f : \$^X \to \$^Y$ s.t. $\exists_f \dashv \$^f$ and $\exists_f(a \sqcap \$^f b) = \exists_{f_!}(a) \sqcap \cancel{\$^f} b$.

X is <u>discrete</u> if $X \xrightarrow{!} 1$ and $\Delta_X : X \to X \times X$ are open; $\mathrm{DIS}_{\mathcal{C}}$ = full-subcategory

For <u>Loc</u> usual notions are recovered.

Recall, $\forall$ topos $\mathcal{E}$,

$$\mathcal{E} \simeq \mathrm{DIS}_{\underline{\mathrm{Loc}}_{\mathcal{E}}} \hookrightarrow \underline{\mathrm{Loc}}_{\mathcal{E}}.$$

'Nice' general categorical result about double exponentiation:

Proposition: If $\mathcal{C}$ is a category with finite products and G an internal group. Given an object A s.t. A^{A^X} exists $\forall X$, then $(A, \pi_2)^{(A,\pi_2)^{(X,a)}}$ exists in $[G, \mathcal{C}]$ for each G-object (X, a).

G-objects & G-homomorphisms.

e.g. $(X, a: G \times X \to X)$

Proof We show $(A,\pi_2)^{(A,\pi_2)^{(X,a)}} = \left(\mathbb{P}X,\ G\times\mathbb{P}X \xrightarrow{t_{G,X}} \mathbb{P}(G\times X) \xrightarrow{\mathbb{P}a} \mathbb{P}X\right)$

t = strength on $\mathbb{P}$.

Need to show $\forall (Y,b)$

$$[G,\mathcal{C}]\Big((Y,b),\big(\mathbb{P}X,\ \mathbb{P}a\, t_{G,X}\big)\Big) \cong \underline{\mathrm{Nat}}\left[(A,\pi_2)^{(X,a)}, (A,\pi_2)^{(Y,b)}\right]$$

$$\underline{\mathrm{LHS}} \cong \left\{ Y \xrightarrow{\phi} \mathbb{P}X \;\middle|\; \phi \text{ a } G\text{-hom.}\right\} \cong \left\{ \alpha : A^X \to A^Y \;\middle|\; \begin{tikzcd} A^X \arrow[r, "\alpha"] \arrow[d, "A^a"'] & A^Y \arrow[d, "A^b"] \\ A^{G\times X} \arrow[r, "\alpha^G"'] & A^{G\times Y} \end{tikzcd} \right\}$$

exp. transpose

But

$$(A,\pi_2)^{(Y,b)} \xhookrightarrow{A^b} (A,\pi_2)^{(G\times Y,\ m\times \mathrm{Id}_Y)} \underset{A^{\mathrm{Id}\times b}}{\overset{A^{m\times \mathrm{Id}_Y}}{\rightrightarrows}} (A,\pi_2)^{(G\times G\times Y,\ m\times \mathrm{Id}_{G\times Y})}$$

an equalizer (apply $(A,\pi_2)^{(-)}$ to U-split coequalizer $G\times G\times Y \underset{\mathrm{Id}_G\times b}{\overset{m\times \mathrm{Id}_Y}{\rightrightarrows}} G\times Y \xrightarrow{b} Y$)

$$\Rightarrow \underline{\mathrm{RHS}} \hookrightarrow \underline{\mathrm{Nat}}\left[(A,\pi_2)^{(X,a)}, (A,\pi_2)^{(G\times Y,\ m\times \mathrm{Id}_Y)}\right] \cong \underline{\mathrm{Nat}}\left[A^{U(X,a)}, A^Y\right] = \underline{\mathrm{Nat}}\left[A^X, A^Y\right] \qquad \square$$

$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "G\times(-)"] \arrow[r, phantom, "\perp"] & {[G,\mathcal{C}]} \arrow[l, shift left, "U"] \end{tikzcd}$ sat. Frobenius rec. $\Rightarrow$ extends $A^{U(-)} \dashv (A,\pi_2)^{G\times(-)}$

<u>Certainly</u>: $\mathcal{C}$ order enriched $\Rightarrow$ $[G, \mathcal{C}]$ order enriched

$\mathcal{C}$ finite products $\Rightarrow$ $[G, \mathcal{C}]$ has finite products

$\exists$ \$ dlat in $\mathcal{C}$ $\Rightarrow$ $(\$, \pi_2)$ DLAT in $[G, \mathcal{C}]$

$\vdots$ $\Rightarrow$ $\vdots$

category of spaces

can extend G-stability to axioms, so that $\mathcal{C}$ captures a number of key results from locale theory
e.g. open/proper stability, Hof. Mislove ...

So by Prop. double exponentiability also G-stable:

<u>Theorem</u> $\mathcal{C}$ a category of spaces. Then so is $[G, \mathcal{C}]$ for any internal group G. $\square$

<u>Define</u> BG = full subcategory of $[G, \mathcal{C}]$ consisting of (X, a) s.t. X discrete.

Well known (or show axiomatically ...) $\forall$ localic groups: $\mathrm{DIS}_{[G, \underline{\mathrm{Loc}}]} \simeq BG$.

But:

Prop. There exist category of spaces that are not categories of locales.

Proof G an open localic group. Say $[G, \underline{Loc}] \simeq \underline{Loc}_{\mathcal{E}}$ some elementary topos $\mathcal{E}$.

Then,

$$\begin{tikzcd}
{[G, \underline{Loc}]} \arrow[r, phantom, "\simeq"] & \underline{Loc}_{\mathcal{E}} \arrow[r, phantom, "\simeq"] & \underline{Loc}_{BG} \arrow[r, phantom, "\simeq"] & {[\hat{G}, \underline{Loc}]} \\
\underline{OIS}_{[G,\underline{Loc}]} \arrow[u, hook] \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr\!\wr"] & \mathcal{E} \arrow[u, hook] & & \\
BG \arrow[ur, "\simeq"'] & & &
\end{tikzcd}$$

same topos
$\Rightarrow$ same cat. of locales.

$$\underbrace{\begin{tikzcd}
Sh(\hat{G}) \arrow[r] \arrow[d] & \underline{Set} \arrow[d, "q"] \\
\underline{Set} \arrow[r, "q"] & BG
\end{tikzcd}}_{\text{p.b. in } \underline{\text{Toposes}}}$$

q open surjection
$\Rightarrow$ effective descent morphism.

This $\simeq$ (between $[G, \underline{Loc}]$, $[\hat{G}, \underline{Loc}]$) is under $\underline{Loc}$ (i.e. commutes with $u \dashv G \times (-)$, $u \dashv \hat{G} \times (-)$)

and so $G \cong \hat{G}$ (i.e. G étale complete). Take any non-étale complete to finish. $\square$

Defn $\mathcal{C}$ by $\mathrm{Ob}(\mathcal{C})$ = categories of spaces.

$$\mathcal{M}(\mathcal{C}) = \{ L \dashv R : \mathcal{D} \rightleftarrows \mathcal{C} \mid R \text{ preserves } +,\ \exists \text{ iso. } R\,\mathbb{P}_{\mathcal{C}} \cong \mathbb{P}_{\mathcal{D}}\,R \}$$

Then $\exists$ $\underline{\text{Toposes}} \longrightarrow \mathcal{C}$ full and faithful: $\underline{\underline{\text{not}}}$ essentially surjective.

$$\begin{tikzcd}
\mathcal{E} \arrow[r, mapsto] \arrow[d] & \underline{\mathrm{Loc}}_{\mathcal{E}} \arrow[d, shift right=2, "\Sigma_f"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{F} \arrow[r, mapsto] & \underline{\mathrm{Loc}}_{\mathcal{F}} \arrow[u, shift right=2, "f^*"']
\end{tikzcd}$$