Axiomatic Localic Relational Composition

Christopher Townsend (cft71@hotmail.com)

Aim: To give an axiomatic account of the process of forming a locale whose opens are the upper closed sublocales of a localic poset.

Hope: ... an axiomatic account of both Priestley duality and information system representation.

Locale Theory $\underline{\mathrm{Loc}} \equiv \underline{\mathrm{Frm}}^{op}$

Frame $= \bigvee \wedge$, with $\bigvee\{t \wedge a \mid t \in T\} = \bigvee T \wedge a$.

Frame homomorphism = preserves $\bigvee, \wedge$.

Preframe hom. = preserves $\bigvee^{\uparrow}, \wedge$.

Suplattice hom. = preserves $\bigvee$ $(= \bigvee^{\uparrow}, \vee)$

Frames are denoted ΩX, where X is the corresponding locale.

$\underline{\mathrm{Loc}}$ "behaves like" $\underline{\mathrm{Top}}$

EASY EXAMPLE : DISCRETE LOCALIC POSET

FOR ANY SET, X (I.E. DISCRETE LOCALE)

$$\mathrm{Sub}_{\mathrm{DISLOC}}(X \times X) \cong \mathrm{SUPLATTICE}(\Omega X, \Omega X)$$

$$R \longmapsto \alpha_R$$

WHERE $\alpha_R(I) = I \circ R$ RELATIONAL COMPOSITION.

NOTE α TAKES RELATIONAL COMPOSITION TO FUNCTION COMPOSITION.

HENCE IF $(X, \leq)$ IS A POSET, $\alpha_R(I)$ = UPPER CLOSURE OF I.

SO...

$$\mathcal{A}X \longleftrightarrow \Omega X \underset{\alpha_R}{\overset{\mathrm{Id}}{\rightrightarrows}} \Omega X$$

ALEXANDROV OPENS

... SET OF UPPER CLOSED SUBSETS = THIS EQUALIZER

NOW, $\leq \circ \leq = \leq$ SO $\alpha_{\leq}^2 = \alpha_{\leq}$; SPLITTING IDEMPOTENT.

* THIS IS ONE HALF OF (SCOTT'S) INFORMATION SYSTEM REPRESENTATION,

$$\mathcal{A}X \equiv \Omega |\mathrm{Id}| X .$$

HARDER MOTIVATING EXAMPLE

FOR ANY COMPACT HAUSDORFF X,

$$\begin{array}{ccc} \mathrm{Sub}_{\mathrm{KHAUS}}(X \times X) & \xrightarrow{\alpha} & [\mathrm{CLOSED}(X), \mathrm{CLOSED}(X)] \\ R & \longmapsto & (I \longmapsto I \circ R) \end{array}$$

IN TERMS OF <u>OPENS</u>, $\Omega X \cong \mathrm{CLOSED}(X)^{op}$,

$$\mathrm{Sub}_{\mathrm{KHAUS}}(X \times X)^{op} \xrightarrow{\cong} \mathrm{PRE\ FRAME}(\Omega X, \Omega X)$$

FOR EXAMPLE, $(X, \leq)$ A COMPACT HAUSDORFF POSET WE CAN AGAIN SPLIT THE IDEMPOTENT $\alpha_{\leq}$:

$$\Omega(\bar{X}) \hookrightarrow \Omega X \underset{\alpha_{\leq}}{\overset{\mathrm{id}}{\rightrightarrows}} \Omega X$$

$\Omega(\bar{X})$ = STABLY CONTINUOUS FRAME.

$$\boxed{\begin{array}{c} 1 \ll 1 \\ b \ll a_1, a_2 \Rightarrow b \ll a_1 \wedge a_2 \end{array}}$$

I.E. CONTINUOUS POSET

* BANASCHWESKI & BRÜMMER: ALL STABLY CTS FRAMES ARISE IN THIS WAY.
* EXTENDS PRIESTLEY DUALITY:

$$\begin{array}{ccccc} \underline{\mathrm{OSTONESP}} & \longrightarrow & \underline{\mathrm{COH\ SP}} & \underset{\cong}{\xrightarrow{\mathrm{STONE}}} & \underline{\mathrm{DLAT}}^{op} \\ X & \longmapsto & \bar{X} & & \end{array}$$

LETS LOOK AT THE EXAMPLES AGAIN:

X DISCRETE $\quad \mathrm{Sub}_{\mathrm{DIS}}(X \times X) \cong \mathrm{SUPLATTICE}(\Omega X, \Omega X)$

X COMP. HAUSDORFF $\quad \mathrm{Sub}_{\mathrm{KHAUS}}(X \times X)^{op} \cong \mathrm{PREFRAME}(\Omega X, \Omega X)$

THE SAME???

FACT: PROVIDED WE WORK WITH LOCALES THERE IS A DUALITY BETWEEN THE SUPLATTICE APPROACH AND THE PREFRAME APPROACH. UNDER THIS DUALITY, COMPACT HAUSDORFF TRANSLATES TO DISCRETE.

[VERMEULEN, "PROPER MAPS OF LOCALES" JPAA '92]

[VICKERS "POINTS OF POWER LOCALES" MATH. PROC. CAMB. '97]

[JOHNSTONE "SKETCHES OF AN ELEPHANT"]

TAYLOR'S "ABSTRACT STONE DUALITY" PROGRAMME ALSO ENJOYS THIS DUALITY.

HERE ARE SOME AXIOMS FOR $\underline{\mathrm{Loc}}$:

1. $\mathbb{C}$ ORDER ENRICHED
2. $\mathbb{C}$ HAS FINITE LIMITS/COLIMITS
3. THERE IS $\$ \in \mathrm{DLAT}(\mathbb{C})$ WHICH CLASSIFIES BOTH OPEN AND CLOSED SUBOBJECTS.
4. $\$^{(-)}$ TAKES EQUALIZERS TO (CERTAIN) COEQUALIZERS.

THEN

(A) $\underline{\mathrm{Loc}}$ IS MODEL. $\$$ = SIERPINSKI LOCALE, SO $\underline{\underline{\$^X \sim \Omega X}}$.

(B) AXIOMS STABLE UNDER ORDER DUALITY.

(C) $\mathrm{KHAUS}(\mathbb{C})$ REGULAR. SO IS $\mathrm{Dis}(\mathbb{C})$ SINCE $\mathrm{DIS}(\mathbb{C}) \cong \mathrm{KHAUS}(\mathbb{C}^{co})$.

<u>EXAMPLES AGAIN</u>

$X \in \mathrm{Dis}(\mathbb{C})$ $\quad \mathrm{Sub}_{\mathrm{DIS}(\mathbb{C})}(X \times X) \cong \sqcup\text{-}\mathrm{SLAT}[\$^X, \$^X]$

$X \in \mathrm{KHAUS}(\mathbb{C})$ $\quad \mathrm{Sub}_{\mathrm{KHAUS}(\mathbb{C})}(X \times X)^{op} \cong \sqcap\text{-}\mathrm{SLAT}[\$^X, \$^X]$

<u>NOTE</u> FOR $\mathbb{C} = \underline{\mathrm{Loc}}$ $\quad \mathrm{dcpo}(\Omega X, \Omega X) \cong [\$^X, \$^X]$.

NEW! FOR LOCALES THERE EXISTS UPPER AND LOWER POWER LOCALE CONSTUCTIONS $P_U, P_L : \underline{Loc} \longrightarrow \underline{Loc}$.

BY DEFINITION: FOR ANY LOCALES X, Y:

$$\underline{Loc}(Y, P_U X) \cong \text{PREFRAME}(\Omega X, \Omega Y)$$

$$\underline{Loc}(Y, P_L X) \cong \text{SUPLATTICE}(\Omega X, \Omega Y)$$

SO... NEW AXIOM FOR $\mathbb{C}$

$\boxed{\begin{array}{l} \text{⑤ FOR ALL } X \in \mathbb{C} \ \exists P_U X, P_L X \text{ SUCH THAT} \\ \text{FOR ALL } Y, \\ \mathbb{C}(Y, P_U X) \cong \sqcap\text{-SLAT}[\$^X, \$^Y] \\ \mathbb{C}(Y, P_L X) \cong \sqcup\text{-SLAT}[\$^X, \$^Y] \end{array}}$

THEOREM:

$$X \in Dis(\mathbb{C}) \quad Sub_{DIS(\mathbb{C})}(X \times X) \cong \sqcup\text{-SLAT}[\$^X, \$^X]$$

$$X \in KHAUS(\mathbb{C}) \quad Sub_{KHAUS(\mathbb{C})}(X \times X)^{op} \cong \sqcap\text{-SLAT}[\$^X, \$^X].$$

Outline Proof

S.T.P. $\mathrm{Sub}_{\mathrm{KHAUS}}(X \times X)^{op} \cong \mathbb{C}(X, P_U X)$

SINCE $\mathbb{C}(X, P_U X) \cong \sqcap\text{-SLAT}[\$^X, \$^X]$ BY AXIOM.

Overview: DESCRIBE $\mathbb{C}(1, P_U X)$ THEN EXPLOIT SLICE STABILITY OF AXIOMS: $\mathbb{C}(X, P_U X) \cong \mathbb{C}/_X(1, P_U^* X)$.

Well: $\mathbb{C}(1, P_U X)^{op} \cong \{X_0 \overset{i}{\hookrightarrow} X \mid X_0 \text{ COMPACT}, i \text{ FITTED}\}$

WHERE $i : X_0 \hookrightarrow X$ **FITTED** IF THERE EXISTS $g : X \to Y$, $p : 1 \to Y$ SUCH THAT

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] & X \arrow[rr, "g"] \arrow[dr, "!"'] & \arrow[d, phantom, "\sqcup\!\sqcup"] & Y \\ & & 1 \arrow[ur, "p"'] & \end{tikzcd}$$

IS A LAX EQUALIZER.

I.E. WE RECOVER HOFFMAN/MISLOVE/JOHNSTONE.

BUT X_0 COMPACT PROVIDES,

$$\$^X \xrightarrow{\$^i} \$^{X_0} \xrightarrow{\forall_!} \$ \qquad \sqcap\text{-SLAT},$$

A POINT OF $P_U X$.

CONVERSELY, $p : 1 \to P_U X$ GIVES FITTED,

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] & X \arrow[rr, "\eta_X^U"] \arrow[dr, "!"'] & \arrow[d, phantom, "\sqcup\!\sqcup"] & P_U X \\ & & 1 \arrow[ur, "p"'] & \end{tikzcd}$$

X_0 COMPACT SINCE $\$^{(-)}$ TAKES EQUALIZERS TO (CERTAIN) COEQUALIZERS

//

SO, FOR ANY COMPACT HAUSDORFF POSET $(X, \leq)$
(OR DISCRETE)

IN $\mathbb{C}$:

$$\begin{tikzcd} \$^X \arrow[r, shift left, "id"] \arrow[r, shift right, "\alpha_{\leq}"'] & \$^X \end{tikzcd} \qquad (*)$$

HAS $\alpha_{\leq}$ IDEMPOTENT. (SINCE $\alpha_{\leq \cdot \leq} = \alpha_{\leq} \alpha_{\leq}$.)

GIVEN,

NEW AXIOM: THE FULL SUBCATEGORY OF OBJECTS OF THE FORM $\$^X$ IS CAUCHY COMPLETE,

THERE THEN EXISTS $\$^{\bar{X}}$ SPLITTING $(*)$.

$(X, \leq) \longmapsto \bar{X}$ THEN DEFINES BOTH,

$$\begin{tikzcd} \mathrm{KHAUSPOS}(\mathbb{C}) \arrow[r] & \mathrm{STLOCK}(\mathbb{C}) \arrow[l, dashed, bend left, "?^{u}"] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{DISPOS}(\mathbb{C}) \arrow[r] & \mathrm{ALG\,DCPO}(\mathbb{C}) \arrow[l, dashed, bend left, "?^{L}"] \end{tikzcd}$$

BUT IN PRACTICE $?^{u}$ AND $?^{L}$ VERY DIFFERENT:

$?^{u}$ = PATCH,

$?^{L}$ = SET OF COMPACT POINTS.

Application

Recall for any set X, PX is the free suplattice on X.

By analogy, from the axioms, we have

<u>Lemma:</u> For any $X \in Dis(\mathbb{C})$ any $\ulcorner a \urcorner : X \to \$^\omega$ factors as

$$\begin{tikzcd}
 & \$^X \arrow[dr, "\gamma_{\ulcorner a \urcorner}"] & \\
X \arrow[ur, "\ulcorner = \urcorner"] \arrow[rr, "\ulcorner a \urcorner"'] & & \$^\omega
\end{tikzcd}$$

for unique ω-SLAT $\gamma_{\ulcorner a \urcorner}$. //

I.E. $\$^X$ is (restricted) free suplattice on X.

I.E. "(A tiny bit of) set theory without set theoretic axioms"

SUMMARY

* BY PROVIDING AN AXIOMATIC ACCOUNT OF LOCALES, WE HAVE AN AXIOMATIC ACCOUNT OF THE PROCESS OF FORMING AN OBJECT OF ALL UPPER CLOSED SUBOBJECTS OF A DISCRETE POS-OBJECT.

* COMPACT HAUSDORFF OBJECTS AND DISCRETE OBJECTS FOLLOW IDENTICAL THEORIES AND ARE RELATED BY TAKING ORDER DUALITY.

* THE PROCESS OF TAKING THE OBJECT OF UPPER CLOSURES IS ONE HALF OF <u>BOTH</u> INFORMATION SYSTEM REPRESENTATION AND (EXTENDED) PRIESTLEY DUALITY. <u>IF</u> THE INVERSE FUCTOR COULD BE DESCRIBED AXIOMATICALLY THEN BOTH THESE REPR. THEOREMS COULD BE DESCRIBED VIA THE SAME ABSTRACT RESULT.